Integration of Probabilistic Methodology in the Aerospace Design Optimization Process

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The need of improvements in engineering designs especially with composite materials is nowadays a major request of the aerospace industry. Deterministic approaches are unable to take into account all the variabilities that characterize composites properties without leading to oversized structures. This paper intends to give a description of the most commonly used methods for reliability-based design optimization using the reliability index and the performance measure approaches to point out the advantages of the application of these methods in the design process. For this purpose, a Finite Element model has been created for an industrial composite wing structure and realistic variability has been assigned to material properties and static load conditions. A range of test cases with gradually increasing complexity has then been defined, that allows the assessment of reported methods in terms of accuracy, computation time and applicability in conjunction with Finite Element models.

Nomenclature

\( E \) = Young’s modulus
\( F_s \) = Cumulative Distribution Function (CDF)
\( G_{12} \) = Shear modulus
\( G_j \) = \( j^{th} \) Performance Function
\( U \) = Stochastic standard normal space
\( X \) = Stochastic input parameters space
\( f_x \) = Probability Density Function (PDF)
\( g_j \) = Performance measure
\( \Phi \) = Standard Normal Cumulative Distribution Function
\( \Omega_j \) = Stochastic subspace of failure events w.r.t \( j^{th} \) performance function
\( \beta_t \) = Target reliability index
\( \beta_s \) = Estimated reliability index
\( \rho \) = Density

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I. Introduction

KNOWING the inherent risk of failure in structural design is becoming increasingly important to both the manufacturer and the customer. Unfortunately current aerospace design analysis methods do not directly account for the random nature of the input parameters. This is particularly true for composite materials. Inherent variabilities in the manufacturing and assembling processes are not always known or properly characterized. It is however clear that new aircraft developments (e.g. re usable launch vehicle, high speed civil transport) can only be successful if consistent changes in the traditional design procedures are made.

From this point of view, it becomes vital to better understand how scatter in physical properties affects the behavior of a structure and to assess which design parameters are most critical. Especially for composites, material defects introduced by the ply manufacturing and by the assembly processes play an important role as well. It is a complex task to quantify these defects in a probabilistic manner, so that a reliability analysis to predict their effect on the structural performance is not always possible. This is mainly due to the lack of accurate characterization of parameter statistics and to the lack of reliable mathematical models.

When accurate stochastic descriptions of physical properties are available, these can be integrated into the design process of composite structures and a procedure management and optimization tool can be developed to

- assess the inherent risk of the preliminary design
- identify the most relevant parameters
- optimize the structure to meet targets on structural performance and safety

In this paper, two approaches will be used to demonstrate the effectiveness of the probabilistic approach. They will be introduced in section III.

II. Test case: the composite structure

To explain the methodology used to assess the design risk, an industrial test case is considered in this paper. A static analysis is performed on a real prototype composite wing for a high altitude, long endurance (HALE) unmanned air vehicle (UAV), for which the characteristics are briefly summarized in Figure 1.

The composite wing has been made of AS4 12k/3502 uniaxial. Many properties of this material have already been statistically characterized in the Military Handbook 17-2E [1] and can therefore immediately be used in a probabilistic formulation [2].

The considered input parameters are

- \(E_1\) and \(E_2\), the Young modulus along the fiber direction and perpendicular to the ply, respectively;
- \(G_{12}\), the shear modulus;
- \(\rho\), the mass density of the composite material.

For the reader’s convenience, they are summarized in Table 1:

<table>
<thead>
<tr>
<th>Property</th>
<th>Distribution</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber (E_1)</td>
<td>Normal</td>
<td>1.3307\times10^{11} [Pa]</td>
<td>7.6381\times10^{9} [Pa]</td>
</tr>
<tr>
<td>Fiber (E_2)</td>
<td>Normal</td>
<td>9.3079\times10^{9} [Pa]</td>
<td>3.9652\times10^{8} [Pa]</td>
</tr>
<tr>
<td>Fiber (G_{12})</td>
<td>Normal</td>
<td>3.7438\times10^{9} [Pa]</td>
<td>1.9318\times10^{8} [Pa]</td>
</tr>
<tr>
<td>Density (\rho) (Rho)</td>
<td>Normal</td>
<td>1575 [Kg/m^3]</td>
<td>2.5 [Kg/m^3]</td>
</tr>
<tr>
<td>Stringer Fiber (E_1)</td>
<td>Normal</td>
<td>1.3307\times10^{11} [Pa]</td>
<td>7.6381\times10^{9} [Pa]</td>
</tr>
</tbody>
</table>

Table 1. Statistically characterized material properties at 75\(^\circ\) F (23.89\(^\circ\) C)

To stiffen the structure, stringers are distributed over the entire wing; they are considered as one-dimensional elements in the Finite Element (FE) model, so only the fiber modulus along the fiber direction has been used (Stringer \(E_1\), SE). It should be noted that in this paper all the variables in Table 1 are considered to be statistically independent. This approximation is not always true in reality. Unfortunately the realistic characterization of the correlation coefficients is often very difficult and therefore considered impractical for most engineering applications.
The composite wing has been examined under gust load conditions while the wing is being clamped in the section that connects to the fuselage. The Pratt formula has been used to evaluate the gust load; it should be noted, however, that a stronger gust situation is considered for the presented cases. To perform the analysis, a Finite Element model of the wing has been created and a particular framework for the necessary computations has been set up. Particularly, the NASTRAN [3] solver has been used for the FE computations and a connection between MATLAB [4] and OPTIMUS [5] has been established for the necessary computations of the reliability-based design optimization algorithms. More details are given in section IV.

III. The reliability problem statement

As stated in section I, the variabilities of an engineering design can be characterized by the variations of a random system parameter set

\[ x = [x_i]^T \quad \text{with} \quad i = 1...n \]

The probability distribution of each \( x_i \) is described either by its cumulative distribution function (CDF, \( F_{x_i}(x_i) \)) or by its probability density function (PDF, \( f_{x_i}(x_i) \)) and is often bounded by tolerance limits on the system parameter values.

The system performance is described by Performance Functions (PF, \( G_j(x) \) with \( j = 1...m \)) that, for a structural design, are usually the selected failure criteria. Thus, with \( G_j(x) \) one of the \( m \) system PFs, the system is considered to fail if \( G_j(x) < 0 \) for at least one index \( j \). The probability of failure of the system for every \( j^{th} \) performance function is then

\[ F_{G_j}(0) = P(G_j(x) < 0) = \int_{\Omega_{G_j}} \ldots \int_{\Omega_{G_j}} f_X(x) dx_1 \ldots dx_n \quad \text{where} \quad \Omega_{G_j} = \{x \in \mathbb{R}^n : G_j(x) < 0\} \quad (1) \]

The event space \( \Omega_{G_j} \) is the region of the stochastic space where only failure events occur. Thus, the integral of the joint probability density function of the system response over \( \Omega_{G_j} \) yields the probability of failure \( P_{f,j} \) of the structure for the \( j^{th} \) criterion. In general, given a fixed performance index (or measure) \( g_j \) of the system, Eq (1) can be generalized

\[ F_{G_j}(g_j) = P(G_j(x) < g_j) = \int_{\Omega_{G_j}} \ldots \int_{\Omega_{G_j}} f_X(x) dx_1 \ldots dx_n \quad \text{where} \quad \Omega_{G_j} = \{x \in \mathbb{R}^n : G_j(x) < g_j\} \quad (2) \]

In this case, the event space \( \Omega_{G_j} \) represents the region of the stochastic space where the performance of the structure is below a prescribed quantity \( g_j \).

A. The Reliability-Based Design Optimization (RBDO) Model

In engineering design, the traditional deterministic design optimization model has been successfully applied to systematically improve the system design process, yielding a reduction of the costs and an improvement of the final quality of the products. However, uncertainties in either engineering simulations and/or manufacturing processes exist. This calls for different optimization models that can yield not only an improvement in the design, but also a higher level of confidence. Thus, a reliability-based design optimization (RBDO) model for robust and cost-effective designs can be defined using mean values of the random system parameters as design variables and optimizing the cost subject to prescribed probabilistic constraints (like probabilities of failure) by solving a mathematically nonlinear programming problem. As a result, the RBDO solution provides not only an improved design but also a higher level of confidence in the design.

The general RBDO model can be defined as

\[
\begin{align*}
\min_{d} & \text{Cost}(d) \\
\text{s.t.} & \, P_{f,j} = P(G_j(x) < 0) \leq \bar{P}_{f,j} \quad \text{with} \quad j = 1...m
\end{align*}
\]

where the cost function can be any function of the design variable \( d = [d_i]^T = [\mu_i]^T \) and a probabilistic constraint \( \bar{P}_{f,j} \) can be defined for each failure mode. Thus, for each iteration of the optimization loop, an estimation of the
probabilistic constraint has to be computed. For this purpose, different methods exist. In general, each prescribed probability of failure \( P_{f,j} \) can be represented in terms of the \textit{target reliability index} \( \beta_{i,j} \) as

\[
\beta_{i,j} = -\Phi^{-1}(P_{f,j}) = \Phi(-\beta_{i,j}) \tag{4}
\]

also defined by the more general relation

\[
F_{G_i}(g_j) = P(G_j(x) < g_j) = \Phi(-\beta_{i,j}) \tag{5}
\]

where \( \Phi(*) \) is the standard normal CDF (zero mean and standard variation 1).

Using Eq. (5), the second condition of Eq. (3) can be rewritten as

\[
P_{f,j} = G_{i}(0) = \Phi(-\beta_{i,j}) \Rightarrow \beta_{s,j} \geq \beta_{i,j} \tag{6}
\]

where \( \beta_{s,j} \) is traditionally called “reliability index” of the structure for the specified failure mode \( j \). Through inverse transformations, Eq (6) can be written as

\[
g_{j}^{*} = F_{G_i}^{-1}(\Phi(-\beta_{i,j})) \geq 0 \tag{7}
\]

where \( g_{j}^{*} \) is named “target probabilistic performance measure” and represents the value of the performance function “equivalent” to the target reliability index \( \beta_{i,j} \).

\section*{B. Reliability Index Approach (RIA)}

The formulation of the general RBDO model of Eq. (3) that uses Eq. (6) to describe the probabilistic constraint is called Reliability Index Approach (RIA) and can be re-written in the form

\[
\min_{d} \left[ \text{Cost}(d) \right] \quad \text{s.t.} \quad \beta_{i,j}(d) \geq 0 \quad \text{with} \quad j = 1, \ldots, m \tag{8}
\]

At a given design \( d^{(k)} = \left[ d_i^{(k)} \right]^T = \left[ \mu_i^{(k)} \right]^T \) for the \( k^{th} \) iteration of the optimization loop, the evaluation of the reliability index \( \beta_{i,j}(d^{(k)}) \) is performed using

\[
\beta_{i,j}(d^{(k)}) = -\Phi^{-1}\left( \int_{\Omega_j} \ldots \int f_X(x)dx_1 \ldots dx_n \right) \tag{9}
\]

For each iteration \( k \) of the optimization process the following problem has to be solved

\[
\min_{a} \left[ \| a \| \right] \quad \text{s.t.} \quad G_j(a) = 0 \quad \text{with} \quad j = 1, \ldots, m \tag{10}
\]

The approximate solution of this problem yields an estimation of the reliability index as

\[
\beta_{i,j} \approx \left\| a_{G_j=0} \right\| \tag{11}
\]

With this formulation, an estimation of the reliability index \( \beta_{s,j} \) has to be done for each iteration, depending on the design point, and the target of the optimization is to find the design point that satisfies all the probabilistic constraints in terms of the reliability index.

\section*{C. Performance Measure Approach (PMA)}

Another way to formulate the general RBDO model is to use the inverse formulation of the probabilistic constraint of Eq. (7) [6]. The RBDO model can be re-written as

\[
\min_{d} \left[ \text{Cost}(d) \right] \quad \text{s.t.} \quad g_{j}^{*} = F_{G_i}^{-1}(\Phi(-\beta_{i,j})) \geq 0 \quad \text{with} \quad j = 1, \ldots, m \tag{12}
\]

In this case, at a given design \( d^{(k)} = \left[ d_i^{(k)} \right]^T = \left[ \mu_i^{(k)} \right]^T \), the evaluation of the target probabilistic performance measure \( g_{j}^{*} \) is performed using
For each iteration of the optimization process the following problem has to be solved

\[
\min_a \left[ G_j(u) \right] \quad \text{with} \quad j = 1 \ldots m \\
\text{s.t.} \quad \|u\| = \beta_j
\]

(14)

The approximate solution of this problem yields an estimation of the performance measure as

\[
g^*_j \approx \left\| G(u_{\beta=\beta_j}) \right\|
\]

(15)

With this formulation, the performance measure \( g^*_j \), which depends on the design points, has to be estimated within each iteration and the target of the optimization is to find the design point that can satisfy all the probabilistic constraints in terms of the performance measure.

D. Limit State Approximations (LSA)

In most industrial cases, the integrals of Eq. (9) and or Eq. (13) cannot be evaluated in closed form. Only for simple academic cases \( f_\Omega \) can be defined analytically. In such a case, the performance function \( G_j(x) \) for each specific failure mode of the structure is available so that \( \Omega_j \) can be defined. The boundary of the domain \( \Omega_j \) is also called limit-state surface and the function \( G_j(x) \) is called the limit-state function (LSF) [7].

In the most widely used reliability method, approximations are made in the space of standard uncorrelated normal variates, \( Y \), obtained from a transformation of the basic variables

\[
Y = T(X)
\]

(16)

where the transformation \( T \) is expressed in terms of the distributions of \( X \).

In the space \( Y \), denoted as the standard normal space, approximations of the probability of failure \( P_{f,j} \) of Eq. (3) are obtained by replacing the limit state surface with first or second order approximating surfaces. These surfaces are fitted to the limit state surface at points with minimal distance to the origin (design points).

For the sake of simplicity, only the case with one failure criterion \((j=1)\) will be considered in this paper, therefore the \( j \) index is removed from all the quantities that depend on \( j \).

E. Limitations of LSA

Although they are well established and widely used, LSA approximations such as the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) exhibit some limitations when the degree of nonlinearity in the LSF increases. Since the FORM approach is a linear approximation, it is easy to understand that it ignores all nonlinearities in the LSF. As a result, it only relies on the estimated reliability index \( \beta \). Also, note that if the LSF is defined as a series of failure criteria (union of failure domains) or as parallel systems (intersection of failure domains) special extensions should be considered [8]. Furthermore, the SORM approximation needs the evaluation of the main curvatures, which makes it not suitable for high dimensional problems and multiple failure domains [9].

As the number of dimensions of the problem increases, also the error of the approximation for all Limit State Approximations increases accordingly, without the possibility to estimate this error. This means that for problems with a number of dimensions \( n>15 \), special care should be paid to the degree of nonlinearity of the performance function(s) because the error of the approximation strongly depends on it and cannot be known nor estimated [9].

For large dimension \((n>30)\) analysies, also the performance of gradient-based algorithms, as used in the HL-RF algorithm and in PMA procedures, decreases as the dimension of the problem increases. Furthermore the computation time of finite difference gradients for high dimensional problems also depends on the increased number of points that is needed to accurately model the non-linearity.

Finally, the necessity to compute the position of the MPP using a gradient-based algorithm does not guarantee that the obtained point is a global minimum, especially when non-normal distributions are used. Therefore, special procedures are needed to deal with nonlinearities, which further increase the computational effort that is required.
F. First and Second Order Reliability Methods

In the first order reliability method (FORM), the limit state surface is replaced with its tangent hyperplane in the standard normal space \( Y \) at the global minimum distance point (Most Probable Point, MPP) on the limit state surface. With this approximation, the reliability index \( \beta \) coincides with the distance of the MPP to the origin. In the second order reliability method (SORM) the limit state surface is approximated by a paraboloid, defined with the same principal curvatures as estimated for the limit state surface at the design point (MPP). Determination of these curvatures essentially requires computing the second-order derivative matrix of the limit state surface and solving an associated eigenvalue problem.

The determination of the MPP involves the solution of the following constrained optimization problem:

\[
\begin{aligned}
\min_y & \quad f(y) = \frac{1}{2} y^T y \\
\text{s.t.} & \quad G(y) = 0
\end{aligned}
\]  

To achieve the optimal solution, the Karush-Kuhn-Tucker (KKT) conditions have to be satisfied. For only one equality constraint, the Lagrangian expression and the KKT conditions can be reduced to:

\[
\begin{aligned}
L(y, \lambda) &= f(y) + \lambda G(y) \\
\nabla_y L(y^*, \lambda^*) &= 0 \\
G(y^*) &= 0
\end{aligned}
\]

where \((y^*, \lambda^*)\) is the solution point.

Many algorithms are available in literature to solve this problem, usually combining a direction search algorithm and a line search procedure along that direction. A well-established algorithm for the reliability index approach (RIA) is the modified HL-RF algorithm, and for the Performance Measure Approach (PMA), the Hybrid Mean Value (HMV) scheme \([10]\) exhibits good performance.

G. Considerations on using the Finite Element Method

All the algorithms used, however, don’t consider if the point they are searching for has a physical meaning or not. They only solve the mathematical problem, thus some restrictions should be enforced to avoid that the MPP is searched in the region of the transformed space where the results have no physical meaning (although they are mathematically possible). In this paper the word “physical” refers to the used FE model and to the particular characterization of the variability of the input parameters, including their boundaries.

When dealing with FE computations, four types of models can be identified:

- **Type 1**: models with physically meaningful parameters that yield accurate results. That is, the combination of input parameters yielding results that are valid within the boundaries of the selected FE analysis theory (e.g. linear, non-linear, buckling, large deformations etc.)
- **Type 2**: models with physically meaningful parameters that yield inaccurate results. In this case, even if physically meaningful input parameters are used, the solver yields inaccurate results (that is, with no physical meaning). For example, a negative limit state surface value denotes that a failure has occurred somewhere in the system (by definition of the limit state surface), but carries no usable information on the actual magnitude of the stress/strain field that produced the failure. The main reason for this behavior is that FE solvers are usually not able to predict fractures and their consequent effects on the structure.
- **Type 3**: models with physically meaningless parameters that yield inaccurate results. Those models usually have values of the input parameters, which are not physically meaningful. Models of this type are generally located very close to the region where no FE computation can be performed.
- **Type 4**: models with physically meaningless parameters that yield no results. This case seems very simple but has some implications.

![Figure 2. Simplified representation of the boundary problem for the impossible event.](image-url)
The points where FE computations are not possible at all represent a boundary to the stochastic domain. No mathematical algorithm should search outside that boundary. This observation immediately recalls the tail sensitivity problem often reported in statistical literature. Here the boundaries mathematically cut the tails of the n-dimensional normal distribution, explicitly assigning a zero probability (impossible event) to the excluded domain. Thus, the effective stochastic space (having total probability equal to 1) is the region inside the boundary between possible and impossible computations (that is, the region with possible computations). Thus the FORM approximation should be corrected, considering that within this domain, the estimated $P_f$ is the integral of the performance function over the domain, defined by all possible computations that do not lie within the ellipse.

Figure 2 shows a simplified sketch of the boundary problem for these four model types [2].

In this paper the above-mentioned problem has been simplified: the input parameters have been bounded inside a hypercube of $20\sigma$ per side centered in the origin. This avoids feeding NASTRAN with Type 4 models and keeps the algorithm inside the region that can be explored by the Finite Element method. For the current problem that is bounded in a hypercube of $20\sigma$, the nearest boundary is located at $10\sigma$ from the origin, which corresponds to a failure probability of less than $2.0\cdot10^{-23}$. In the case of an elliptic LSF, the nearest point of the ellipse could be located inside the hypercube; in that case, truncating the problem region has no adverse effect. It could also be the case that the nearest point lays outside of the hypercube of $20\sigma$; in that case, it is guaranteed that the reliability index is higher than $10\sigma$, which is sufficient for all practical purposes.

IV. Testcase implementation and results

For the composite wing structure proposed in Section II, a Finite Element model has been created and the external load configuration has been applied. With the variability introduced in the input parameters, the performance behavior of the FE model has been observed and the RBDO process reported in Section III has been applied to meet the following probabilistic constraint:

$$\beta_t = 6 \Rightarrow P_f = 9.86^{-10}$$

For this purpose, the methods reported in Sections III.B and III.C have been implemented in a MATLAB code and used to perform the analysis. All these algorithms require the evaluation of the performance function at specific values of the input parameters, so that a NASTRAN static analysis has to be performed for each evaluation. The Optimization process is managed by an OPTIMUS thread (Figure 3) that determines the necessary steps to reach the optimum design point that satisfies the probabilistic constraint of Eq. (20). The optimization algorithm selected for all testcases is the Sequential Quadratic Programming (SQP) using a tolerance of $10^{-3}$ and forward finite difference gradient estimation. Note that the optimization process is a deterministic process with a probabilistic constraint. Thus the result is an optimum point that is also robust with respect to the
required failure probability. For each iteration of the optimization process, one requires an evaluation of the performance function or of the reliability index, depending on the particular problem formulation (PMA or RIA, respectively). To accomplish this task, another process management thread (Figure 4) using OPTIMUS has been created: when the MATLAB algorithm (for the estimation of the reliability index or of the performance measure) requires a performance function evaluation for a given parameter vector, OPTIMUS launches a NASTRAN computation and submits the results back to MATLAB, where the LSF criterion is evaluated. A simple outline of the whole RBDO process, both for RIA and PMA, is given in Figure 5.

H. Performance Function Choice

As the wing considered in this paper consists entirely of composite material, a special failure criterion should be chosen. Selecting a valid failure theory for a composite material is not straightforward: it should always be decided which composite modeling theory is considered and which failure criteria are provided by the selected theory. Each lamina of the laminates used in the wing structure is composed of strong continuous fiber (stiff along their length) merged in a flexible matrix and supported by a honeycomb core. Thus the fibers react to various load conditions in different ways. Moreover, the fiber angle can slightly change from lamina to lamina in the same laminate. As a result, there exist more failure modes for composites than for metallic materials. The majority of failure criteria for composite materials are based on energetic theories, but few of them, like the Tsai-Wu criterion, have been developed and tested for industrial composite applications. It is common convention to consider that the composite structure breaks when one of its constituents breaks, as until now there are no criteria that take into account the presence of residual stress in the laminate. In the case considered in this paper the Tsai-Hill failure criterion, derived from the more general Hill criterion for anisotropic materials, has been used as performance function. This criterion is defined by the formula:

\[
G = 1 - \left[ \left( \frac{\sigma_1}{F_1} \right)^2 - \frac{\sigma_1 \sigma_2}{F_1 F_2} + \left( \frac{\sigma_2}{F_2} \right)^2 + \left( \frac{\sigma_{12}}{F_{12}} \right)^2 \right]^{1/2}
\]

The values of the Tsai-Hill criterion are directly obtained from NASTRAN for each single ply of each laminate. The maximum Tsai-Hill index over all the plies of all laminates (i.e. a first-ply failure criterion) represents the performance of the structure. An example of the resulting multidimensional surface, given the input parameter variability, is represented in Figure 6: the displayed values belong to a section of the multidimensional space referring to the main hyper-planes passing through the origin. To prevent having to deal with an infinite design space, the transformed standard normal space is bounded in a hyper-cube between \(-10\sigma\) and \(+10\sigma\) and the optimization range for each variable is between \(-4\sigma\) and \(+4\sigma\) with respect to the nominal starting value (these are the initial mean values of the parameters exhibiting variability). As can be seen in Figure 6, the surface is quite flat around the nominal values of the design parameters and becomes quite steep (w.r.t. the scatter around the nominal value) when approaching the LSF.
Four test cases have been designed, using either the input variability in Table 1 or the nominal values. The test cases follow the description given in Table 2.

<table>
<thead>
<tr>
<th>Testcase</th>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
<th>Variable 4</th>
<th>Variable 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fiber E₁</td>
<td>Fiber E₂</td>
<td>Mean Fiber G₁₂</td>
<td>Mean ρ (Rho)</td>
<td>Mean Stringer E₁</td>
</tr>
<tr>
<td>2</td>
<td>Fiber E₁</td>
<td>Fiber E₂</td>
<td>Fiber G₁₂</td>
<td>Mean ρ (Rho)</td>
<td>Mean Stringer E₁</td>
</tr>
<tr>
<td>3</td>
<td>Fiber E₁</td>
<td>Fiber E₂</td>
<td>Fiber G₁₂</td>
<td>ρ (Rho)</td>
<td>Mean Stringer E₁</td>
</tr>
<tr>
<td>4</td>
<td>Fiber E₁</td>
<td>Fiber E₂</td>
<td>Fiber G₁₂</td>
<td>ρ (Rho)</td>
<td>Stringer E₁</td>
</tr>
</tbody>
</table>

Table 2. Definitions of Test cases.

In this paper, only material properties have been considered. Note however that the methodology is not limited to this class of parameters as similar analyses can be performed with geometrical or strength properties. The increasing number of parameters in the test case sequence in Table 2 will be used to show how the addition of new parameters influences the results of the optimization procedure (and the corresponding reliability index β for each iteration).

I. Results

The results of the two optimization processes are here reported and compared. Table 3 shows the results for the optimization using RIA and Table 4 shows the results of the same optimization using PMA. Both the optimization processes have been carried out for all 4 testcases, in order to demonstrate how the results may change by introducing additional variability.

<table>
<thead>
<tr>
<th>SQP Iteration</th>
<th>Reliability Index βₜ</th>
<th>TC 1</th>
<th>TC 2</th>
<th>TC 3</th>
<th>TC 4</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>9.8704</td>
<td>9.8719</td>
<td>9.8487</td>
<td>5.7127</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6.9811</td>
<td>6.0429</td>
<td>7.8666</td>
<td>5.9639</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.7794</td>
<td>6.0192</td>
<td>5.874</td>
<td>5.9752</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6.0057</td>
<td>-</td>
<td>6.5542</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>6.5541</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Reliability Index estimation during the optimization for different testcases using the RIA (results in the table are expressed as multiples of the standard deviation σ in the standard space)

<table>
<thead>
<tr>
<th>SQP Iteration</th>
<th>Performance Measure G(x)</th>
<th>TC 1</th>
<th>TC 2</th>
<th>TC 3</th>
<th>TC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2514</td>
<td>0.2523</td>
<td>0.2523</td>
<td>-0.044</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.1182</td>
<td>0.1252</td>
<td>0.1117</td>
<td>0.0046</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0441</td>
<td>0.1242</td>
<td>0.1036</td>
<td>0.0025</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.0034</td>
<td>0.0615</td>
<td>0.0695</td>
<td>-0.0009</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0003</td>
<td>0.0163</td>
<td>0.021</td>
<td>-0.0005</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0001</td>
<td>0.0054</td>
<td>-0.0108</td>
<td>0.0001</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>0.0049</td>
<td>0.0031</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>0.0046</td>
<td>0.0023</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>0.0033</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Performance Measure estimation in the optimization results for different testcases using PMA (results in the table are expressed in terms of performance function value in the parameter space)

The rate of convergence for the SQP algorithm seemed to be a bit slow, but this was due to the quite steep gradient of the Limit State Surface in the proximity of the LSF. Note also that the results of the Tsai-Hill indices computed by NASTRAN were defined with only 4 decimal digits. The estimation of the gradient along a direction...
might not be very accurate due to the limited number of decimals, yielding a slow convergence rate. To reduce this effect, a semi-adaptive step for the finite difference gradient estimation has been used.

A visualization of the optimization process is shown in Figure 7(a) and Figure 7(b) for the RIA approach and in Figure 8(a) and Figure 8(b) for the PMA approach.

Table 5 shows that the PMA approach for the RBDO process performs much better compared to RIA, not only when one considers the number of LSF evaluations required to find the optimum point, using the same optimization parameters for tolerance and gradient estimation, but also in terms of the quality of the results. The optimum points found are reported in Table 6.

<table>
<thead>
<tr>
<th>Testcase</th>
<th>RIA</th>
<th>PMA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>759</td>
<td>630</td>
</tr>
<tr>
<td>2</td>
<td>2572</td>
<td>2076</td>
</tr>
<tr>
<td>3</td>
<td>3906</td>
<td>1775</td>
</tr>
<tr>
<td>4</td>
<td>2315</td>
<td>3605</td>
</tr>
</tbody>
</table>

Table 5. Optimization performance comparison in terms of Limit State Surface evaluations

In fact, the positions of the optimum points appear to be quite different for the two approaches. It has been tested that the quality of the estimation of the reliability index in the RIA approach for the computation of the gradient is strongly dependent on the pre-defined tolerance. It has been verified that lowering the tolerance from $10^{-3}$ to $10^{-4}$ noticeably increases the number of iterations required to complete both the optimization and the estimation of the reliability index $\beta$. In this refined RIA analysis, the position of the optimal point approaches the location of the
same point computed using PMA with a coarser tolerance. For this refinement study, the results for Testcase 1 have been reported in Table 7. This perhaps surprising effect can easily be explained from the shape of the LSF, the fact that FE computations are involved and the nature of the RIA and PMA algorithms. The LSF in the vicinity of the MPP locus appears to be quite “flat”. With the RIA algorithm, one minimizes a simple cost function subject to a complex constraint. The flat LSF and the limited number of digits for the Tsai-Hill index in the NASTRAN FE results strongly affect the numerical performance of the constraint evaluation. On the other hand, the PMA algorithm minimizes a complex cost function subject to a simple constraint, which gives rise to less numerical problems and therefore a better algorithm performance. This is in line with the experience reported in [6].

In general, however, the results reported in Table 6 show that for both approaches, RIA and PMA, the position of the optimum point is much more affected by the introduction of new random parameters than by the type of approach. These random parameters are representative of the variability of the input parameters and are described by probability distributions. In fact, while in the first testcase Fiber $E_1$ is decreased and Fiber $E_2$ is increased, in testcase 4 the opposite occurs.

<table>
<thead>
<tr>
<th>Testcase</th>
<th>Approach</th>
<th>Fiber $E_1$</th>
<th>Fiber $E_2$</th>
<th>Fiber $G_{12}$</th>
<th>$\rho$ (Rho)</th>
<th>Stringer $E_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>RIA</td>
<td>-3.7739</td>
<td>1.9297</td>
<td>Nominal</td>
<td>Nominal</td>
<td>Nominal</td>
</tr>
<tr>
<td></td>
<td>PMA</td>
<td>-1.0388</td>
<td>3.9200</td>
<td>Nominal</td>
<td>Nominal</td>
<td>Nominal</td>
</tr>
<tr>
<td>2</td>
<td>RIA</td>
<td>-0.79673</td>
<td>4.0259</td>
<td>0.7414</td>
<td>Nominal</td>
<td>Nominal</td>
</tr>
<tr>
<td></td>
<td>PMA</td>
<td>-0.8677</td>
<td>4.0134</td>
<td>-1.3472</td>
<td>Nominal</td>
<td>Nominal</td>
</tr>
<tr>
<td>3</td>
<td>RIA</td>
<td>-3.982</td>
<td>0.99589</td>
<td>-4</td>
<td>-0.25674</td>
<td>Nominal</td>
</tr>
<tr>
<td></td>
<td>PMA</td>
<td>-0.7614</td>
<td>4</td>
<td>1.7929</td>
<td>4</td>
<td>Nominal</td>
</tr>
<tr>
<td>4</td>
<td>RIA</td>
<td>0.035243</td>
<td>-0.13184</td>
<td>0.012564</td>
<td>0.00097183</td>
<td>1.8558</td>
</tr>
<tr>
<td></td>
<td>PMA</td>
<td>0.31719</td>
<td>-0.3487</td>
<td>-0.03564</td>
<td>0.81411</td>
<td>1.7215</td>
</tr>
</tbody>
</table>

Table 6. Optimization results in standard normal space (Y-Space)

<table>
<thead>
<tr>
<th>Testcase</th>
<th>Approach</th>
<th>Fiber $E_1$</th>
<th>Fiber $E_2$</th>
<th>Fiber $G_{12}$</th>
<th>$\rho$ (Rho)</th>
<th>Stringer $E_1$</th>
<th>Stringer $E_2$</th>
<th>$\beta_x$</th>
<th>SQP Iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 w/ tol=10^{-4}</td>
<td>RIA</td>
<td>-2.562</td>
<td>2.9656</td>
<td>Nominal</td>
<td>Nominal</td>
<td>Nominal</td>
<td>Nominal</td>
<td></td>
<td>5.9786</td>
</tr>
</tbody>
</table>

Table 7. Optimization results in standard normal space (Y-Space) with lower tolerance for Testcase 1

In particular, the introduction of the last parameter quite strongly influences the position of the optimum point. As the stringers are stiffener elements in the structure, variability in their properties influences the structure more than the 2nd, 3rd and 4th parameter. This can be better observed in Figure 9 that plots the LSF for the 2nd and 5th parameter.

V. Conclusions

In this paper, a composite wing has been analyzed under static load conditions and with variabilities in the input parameters. Several observations have been made.
With respect to the choice of the input parameters, note that the meaning of optimizing physical quantities is not necessarily straightforward. The change in the physical properties of the material shows what is the effect of using a stiffer or softer material for a specific design, and shows also how a particular combination of both can lead to different designs. This is better shown in Table 6 for testcases 1 and 4 (Fiber E₁) and for testcases 2 and 3 (G₁₂).

With respect to the search for the optimum design point, comparable predictions within the same approach have been obtained for test cases 1 to 3, even with the pre-defined [-4 σ : +4 σ] optimization range defined for the input parameters. The outcome of testcase 4, in terms of reliability index or performance measure for each RIA or PMA computation, is not only influenced by the probability that high- or low-quality materials are used, but also by the probability that a particular combination of different panel and stringer material properties occurs. The effect of nonlinearities introduced by new variabilities in the input parameters should not be underestimated. One should carefully investigate what are the most relevant parameters to be included in the optimization loop through a preliminary assessment of the performance of the structure using RIA and/or PMA.

It has been noticed that RIA is more sensible on the precision selected for the estimation of the reliability index than PMA. Lowering the tolerance of the HL-RF algorithm by one order of magnitude has increased the accuracy of the estimation, but has on the other hand increased the required number of LSF evaluations considerably. This means that RIA requires more iterations to get a result that is comparable with the result obtained with PMA. As already reported in [6] and from what has been explained in section III, it is easier to minimize a complex cost function subject to a simple constraint function than to minimize a simple cost function subject to a complex constrain.

It has been demonstrated that the dimensionality of the problem plays an important role as well. The introduction of the 5th parameter variability drastically changes the configuration of the failure surface defined by the Tsai-Hill criterion. The MPP is pulled closer to the origin and, consequently, the probability of failure is increased. The parameters G₁₂ and ρ have little influence on the performance function and their effect on the overall performance is negligible.

In general, it is possible to say that both approaches are valid for RBDO analysis, but the HL-RF algorithm used for RIA exhibits more sensitivity to the precision of the results and on the particular shape of the LSF function than the HMV algorithm used for PMA. Thus, HMV seems to be less dependent on numerical issues due to complex LSFs than HL-RF, even though the PMA algorithm might sometimes require more computations to converge. Furthermore, PMA seems to be easier to implement for the case of more than one performance function. Thus, PMA with HMV is more suited for RBDO than RIA with HL-RF and it is therefore recommended.

Finally, a remark on the validity of the results obtained with FE computations. They should be trusted only in the feasible domain, but not completely outside this domain. Care should be taken to consider only physically meaningful results and to correctly define both probabilistic distributions and bounded stochastic domain.

Acknowledgments

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References

5. OPTIMUS Rev. 5.0, Noesis Solutions, July 2004.