DESIGN UNDER UNCERTAINTY USING MONTE CARLO SIMULATION AND PROBABILISTIC SUFFICIENCY FACTOR

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ABSTRACT

Monte Carlo simulation is commonly employed to evaluate system probability of failure for problems with multiple failure modes in design under uncertainty. The probability calculated from Monte Carlo simulation has random errors due to limited sample size, which create numerical noise in the dependence of the probability on design variables. This in turn may lead the design to spurious optimum. A probabilistic sufficiency factor (PSF) approach is proposed that combines safety factor and probability of failure. The PSF represents a factor of safety relative to a target probability of failure, and it can be calculated from the results of Monte Carlo simulation (MCS) with little extra computation. The paper presents the use of PSF with a design response surface (DRS), which fits it as function of design variables, filtering out the noise in the results of MCS. It is shown that the DRS for the PSF is more accurate than DRS for probability of failure or for safety index. The PSF also provides more information than probability of failure or safety index for the optimization procedure in regions of low probability of failure. Therefore, the convergence of reliability-based optimization is accelerated. The PSF gives a measure of safety that can be used more readily than probability of failure or safety index by designers to estimate the required weight increase to reach a target safety level. To reduce the computational cost of reliability-based design optimization, a variable-fidelity technique and deterministic optimization were combined with probabilistic sufficiency factor approach. Example problems were studied here to demonstrate the methodology.

Keyword: Design under uncertainty, probabilistic sufficiency factor, Monte Carlo simulation, response surface approximation

INTRODUCTION

Reliability analysis of systems with multiple failure modes often employs Monte Carlo simulation, which generates numerical noise due to limited sample size. Noise in the probability of failure or safety index may cause reliability-based design optimization (RBDO) to converge to a spurious optimum. The accuracy of MCS with a given number of samples deteriorates with decreasing probability of failure. For RBDO problems with small target probability of failure, the accuracy of MCS around the optimum is not as good as in regions with high probability of failure. Furthermore, the probability of failure in some regions may be so low that it is calculated to be zero by MCS. This flat zero probability of failure does not provide gradient information to guide the optimization procedure.

Recently, there has been interest in alternative measures based on margin of safety or safety factors that are commonly used as measures of safety in deterministic design. Safety factor is generally expressed as the quotient of allowable over response, such as the commonly used central safety factor that is defined as the ratio of the mean value of allowable over the mean value of the response. The selection of safety factor for a given problem involves both objective knowledge such as data on the scatter of material properties and subjective knowledge such as expert opinion. Given a safety factor, the reliability of the design is generally unknown, which may lead to unsafe or inefficient design. Therefore, the use of safety factor in reliability-based design optimization seems to be counterproductive.

Freudenthal [1] showed that reliability can be expressed in term of the probability distribution function of the safety factor. Elishakoff [2] surveyed the relationship of safety factor...
and reliability, and showed that in some cases the safety factor can be expressed explicitly in terms of reliability. The standard safety factor is defined with respect to the response obtained with the mean values of the random variables. Thus a safety factor of 1.5 implies that with the mean values of the random variables we have a 50% margin between the response (e.g., stress) and the capacity (e.g., failure stress). However, the value of the safety factor does not tell us what the reliability is. Therefore, Birger [3] introduced a factor, which we call here the probabilistic sufficiency factor (PSF), that is more closely related to the target reliability. A PSF of 1.0 implies that the reliability is equal to the target reliability, a PSF>1 means that the reliability exceeds the target reliability, and PSF<1 means that the system is not as safe as we wish. Specifically, a PSF value of 0.9 means that we need to multiply the response by 0.9 or increase the capacity by 1/0.9 to achieve the target reliability.

If the required maximum probability of failure is \( p_f \), then the probabilistic sufficiency factor is the safety factor exceeded with a probability of \( 1 - p_f \). Consider for example a design with a stress constraint where the target probability of failure is \( 10^{-6} \). A PSF of 1.1 mean that there is a probability of \( 1 - 10^{-6} \) that the stress allowable will be at least 1.1 times larger than the stress.

Tu et al. [4] used a version of Birger’s safety factor, which they called probabilistic performance measure, for RBDO using most probable point (MPP) methods (e.g., first order reliability method). They showed that the search for the optimum design converged faster by driving the safety margin to zero than by driving the probability of failure to its target value. Wu et al. [5, 6] used probabilistic sufficiency factors in order to replace the RBDO with a series of deterministic optimizations by converting reliability constraints to equivalent deterministic constraints.

The use of the probabilistic sufficiency factor gives a designer more quantitative measure of the resources needed to satisfy the safety requirements. For example, if the requirement is that the probability of failure is below \( 10^{-6} \) and the designer finds that the actual probability is \( 10^{-4} \), he or she cannot tell how much change is required to satisfy the requirement. If instead the designer finds that a probability of \( 10^{-5} \) is achieved with a safety factor of 0.9, it is easier to estimate the required resources. For a stress-dominated linear problem, raising the safety factor from 0.9 to 1 typically requires a weight increase of about 10 percent of weight of the overstressed components.

The PSF is readily available from the results of MCS with little extra computational cost. The noise problems of MCS motivate the use of response surface approximation (RSA, e.g., Khuri and Cornell, [7]). Response surface approximations typically employ lower order polynomials to approximate the probability of failure or safety index in terms of design variables in order to filter out noise and facilitate design optimization. These response surfaces are called design response surface (DRS) and are widely used in the RBDO (e.g., Sues et al., [8]).

The probability of failure often changes by several orders of magnitude over narrow bands in design space, especially when the random variables have small coefficients of variation. The steep variation of probability of failure requires DRS to use high-order polynomials for the approximation, increasing the required number of probability calculations (Qu et al., [9]). An additional problem arises when Monte Carlo simulations (MCS) are used for calculating probabilities. For a given number of simulations, the accuracy of the probability estimates deteriorates as the probability of failure decreases.

The numerical problems associated with steep variation of probability of failure led to consideration of alternative measures of safety. The most common one is to use safety index, which replaces the probability by the distance, which is measured as the number of standard deviations from the mean of a normal distribution that gives the same probability. The safety index does not suffer from abrupt changes in magnitude, but it has the same problems of accuracy as the probability of failure when based on Monte Carlo simulations. However, the accuracy of probabilistic sufficiency factor is maintained in the region of low probability. The probabilistic sufficiency factor also exhibits less variation than probability of failure or safety index. Thus the probabilistic sufficiency factor can be used to improve design response surface approximations for RBDO.

**PROBABILISTIC SUFFICIENCY FACTOR APPROACH**

Let \( g(x) \) denote the limit state function of a performance criterion (such as strength allowable larger than stress), so that the failure event is defined as \( g(x) < 0 \), where \( x \) is a random variable. The probability of failure of a system can be calculated as

\[
P_f = \int_{g(x)<0} f_X(x) dx
\]

where \( f_X(x) \) is the joint probability distribution function (JPDF). This integral is hard to evaluate, because the integration domain defined by \( g(x) < 0 \) is usually unknown, and integration in high dimension is difficult. Commonly used probabilistic analysis methods are either moment-based methods such as the first-order-reliability-method (FORM) or the second-order-reliability-method (SORM), or simulation techniques such as Monte Carlo simulation (MCS) (e.g., Melchers, [10]). Monte Carlo simulation is a good method to use for system reliability analysis with multiple failure modes. The present paper focuses on the use of MCS with response surface approximation in RBDO.

The accuracy of MCS depends on the true value of calculated probability, with a small probability requiring a large number of samples for MCS to achieve low relative error. Therefore,
for fixed number of simulations, the accuracy of MCS deteriorates with the decrease of probability of failure. For example, with $10^6$ simulations, a probability estimate of $10^{-3}$ has a relative error of a few percent, while a probability estimate of $10^{-6}$ has a relative error of the order of 100 percent. In RBDO, the required probability of failure is often very low, thus the probability (or safety index) calculated by MCS is inaccurate near the optimum. Furthermore, the probabilities of failure in some design regions may be so low that they are calculated as zero by MCS. This flat zero probability of failure or infinite safety index cannot provide useful gradient information to the optimization.

Here we propose the use of probabilistic sufficiency factor to solve the problems associated with probability calculation by MCS. The deterministic equivalent of reliability constraint can be formulated as

$$g_r, (\hat{x}, d) \leq g_r (\hat{x}, d)$$  \hspace{1cm} (2)

where $g_r$ denotes a response quantity, $g_r$ represents a capacity (e.g., strength limit), $\hat{x}$ is usually the mean value vector of random variables, and $d$ is the design vector. The traditional safety factor is defined as

$$s(x, d) = \frac{g_r (x, d)}{g_r(x, d)}$$  \hspace{1cm} (3)

and the deterministic design problem requires

$$s(\hat{x}, d) \geq s_r$$  \hspace{1cm} (4)

where $s_r$ is the required safety factor, which is usually 1.4 or 1.5 in aerospace applications. The reliability constraint can be formulated as a requirement on the safety factor

$$\text{Prob}(s \leq 1) \leq P_f$$  \hspace{1cm} (5)

where $P_f$ is the required probability of failure. Birger’s probabilistic sufficiency factor $P_{sf}$ is the solution to

$$\text{Prob}(s \leq s_{(n)}) = P_f$$  \hspace{1cm} (6)

It is the safety factor that is violated with the required probability $P_f$. For example, with a stress constraint if the required probability $P_f$ is $10^{-4}$ and $P_f = 1.1$, it means that the probability that the stress limit is below 110% of the stress value is $10^{-4}$. $P_{sf}$ can be estimated by MCS as follows. Define the $n$th safety factor of MCS as

$$s_{(n)} = \frac{n}{M} \min_{i=1}^{n} (s(x_i))$$  \hspace{1cm} (7)

where $M$ is the sample size of MCS, and the $n$th $s_{(n)}$ means the $n$th smallest safety factor among $M$ safety factors from MCS. Thus $s_{(n)}$ is the $n$th-order statistics of $M$ safety factors from MCS, which corresponds to a probability of $n/M$ of $s(x) \leq s_{(n)}$. The probabilistic sufficiency factor is then given as

$$P_{sf} = s_{(n)} \text{ for } n = P_f M$$  \hspace{1cm} (8)

For example, if the required probability $P_f$ is $10^{-4}$ and the sample size of MCS $M$ is $10^5$, $P_{sf}$ is equal to the highest safety factor among the 100 samples ($n = P_f M$) with the lowest safety factors. The calculation of $P_{sf}$ requires only sorting the lowest safety factors in the MCS sample. While the probability of failure changes by several orders of magnitude the probabilistic sufficiency factor usually varies by less than one order of magnitude in a given design space.

For problems with $k$ reliability constraints, the most critical safety factor is calculated first for each MCS sample,

$$s(x_i) = \min_{i=1}^{k} \left( \frac{g_r^i}{g_r} \right)$$  \hspace{1cm} (9)

Then the sorting of the $n$th minimum safety factor can proceed as in (7). When $n$ is small, it may be useful to calculate $P_{sf}$ more accurately as the average between the $n$th and $(n+1)$th lowest safety factor in the MCS samples.

The probabilistic sufficiency factor provides more information than probability of failure or safety index. Even in the regions where the probability of failure is so small that it cannot be estimated accurately by the MCS with given sample size $M$, the accuracy of $P_{sf}$ is maintained. Using the probabilistic sufficiency factor also gives designers useful insights on how to change the design to satisfy safety requirements. For example, in a stress dominated design a probabilistic sufficiency factor of 0.90, i.e., $P_{sf} = 0.9$, means that the stresses need to be reduced by 10%. This often can give the designer an estimate of the required additional weight needed in order to satisfy the reliability constraint, an estimate that is not readily available from the probability of failure. Probabilistic sufficiency factor is based on the ratio of allowable of response, which exhibit much less variation than the probability of failure or safety index. Therefore, approximating probabilistic sufficiency factor in design optimization is easier than approximating probability of failure or safety index as discussed in the next section.

**MONTE CARLO SIMULATION USING RESPONSE SURFACE APPROXIMATION**

MCS is easy to implement, robust, and accurate with sufficiently large samples, but it requires a large number of analyses to obtain a good estimate of small failure probabilities. MCS also produces a noisy result due to limited sample size. Most optimization algorithms are sensitive to noise and may be trapped in spurious optima due to noise.

Response surface approximations solve the two problems, namely simulation cost and noise from random sampling. RSA methods fit a closed-form approximation to the limit state function to facilitate reliability analysis. Therefore, RSA is particularly attractive for computationally expensive problems such as those requiring complex finite element analyses. Response surface approximations usually fit low order polynomials to the structural response in terms of random variables

$$\hat{g}(x) = Z(x)^T b$$  \hspace{1cm} (10)
where \( \hat{g}(\mathbf{x}) \) denotes the approximation to the limit state function \( g(\mathbf{x}) \). \( Z(\mathbf{x}) \) is the basis function vector that usually consists of monomials, and \( \mathbf{b} \) is the coefficient vector estimated by least square regression. The probability of failure can then be calculated inexpensively by MCS or FORM/SORM using the fitted polynomials.

Response surface approximations can be used in different ways. One approach is to construct local response surfaces around the Most Probable Point (MPP) that contributes most to the probability of failure of the structure. The statistical design of experiment (DOE) of this approach is iteratively performed to approach the MPP. For example, Bucher and Bourgund [11], and Sues [8, 12] constructed progressively refined local response surfaces around the MPP by an iterative method. This local RS approach can produce satisfactory results given enough iterations. Another approach is to construct global response surfaces over the entire range of random variables, i.e., DOE around the mean values of the random variables. Fox [13] used Box-Behnken design to construct global response surfaces and summarized 12 criteria to evaluate the accuracy of response surfaces. Romero and Bankston [14] employed progressive lattice sampling as the statistical design of experiments to construct global response surfaces. With this approach, the accuracy of RSA around the MPP is unknown, thus caution must be taken to avoid extrapolation around the MPP. Both approaches can be used to perform reliability analysis for computationally expensive problems. The selection of RSA approach depends on the limit state function of the problem. The global RSA is simpler and efficient to use than local RSA for problems with limit state function that can be well approximated globally.

However, the reliability analysis and hence the RSA needs to be performed at every design point visited by the optimizer. This requires a fairly large number of RS constructions and thus limit state evaluations. The local RS approach is even more computationally expensive than the global approach in the design environment. Qu et al. [9] developed a global analysis response surface (ARS) approach in unified space of random variables, i.e., DOE around the mean values of the random variables. They recommended Latin Hypercube sampling (LHS) as the statistical design of experiments. The number of RSA constructed in optimization process is reduced substantially by introducing design variables into the RSA formulation.

The selection of RS approach depends on the limit state function of the problem and target probability of failure. The global RS approach is simpler than local RS, but it is limited to problems with relatively high probability or limit state function that can be well approximated by regression analysis based on simple basis functions. To avoid the extrapolation problem, RS generally needs to be constructed around important region or Most Probable Point (MPP) to avoid large errors in the results of MCS induced by fitting errors in RS. Therefore, an iterative RS is desirable for general reliability analysis problem.

Design response surface approximations (DRS) are fitted to probability of failure to filter out noise in MCS and facilitate optimization. Based on past experience, high order DRS (such as quintic polynomials) are needed in order to obtain a reasonably accurate approximation of the probability of failure. One difficulty of constructing highly accurate DRS is caused by the fact that the probability of failure changes by several orders of magnitude over small distance in design space. Fitting to safety index \( \beta = \Phi^{-1}(p) \), where \( p \) is the probability of failure and \( \Phi \) is the cumulative distribution function of normal distribution, improves the accuracy of the DRS to a limited extent. Probabilistic sufficiency factor can be used to improve the accuracy of DRS approximation.

**BEAM DESIGN EXAMPLE**

The following cantilever beam example (Fig. 1) is taken from Wu et al. (2001).

![Figure 1](https://example.com/beam_image.png)

**Figure 1. Cantilever beam subject to vertical and lateral beading**

There are two failure modes in the beam design problem. One failure mode is yield, which is most critical at the corner of the rectangular cross section at the fixed end of the beam

\[
R - \sigma = R - \left( \frac{600}{w^2t} \right) Y + \frac{600}{w^2t} X \tag{12}
\]

where \( R \) is the yield strength, \( X \) and \( Y \) are the independent horizontal and vertical loads. Another failure mode is the tip deflection exceeding the allowable displacement, \( D_0 \)

\[
D_t - D = D_0 - 4L^2 \frac{E t}{w^2} \sqrt{\left( \frac{Y}{t} \right)^2 + \left( \frac{X}{w^2} \right)^2} \tag{13}
\]

where \( E \) is the elastic modulus. The random variables are defined in Table 1.

Since the limit state of the problem is available in closed form, the direct MCS approach with a sufficient large number of samples is used here (without analysis response surface) in order to in order to better demonstrate the advantage of PSF over probability of failure or safety index better. By using the exact limit state function, the errors in the results of MCS are
purely due to the convergence errors, which are controllable by changing the sample size of MCS. In applications where ARS approximation must be used, the errors introduced by approximation can be reduced by sequentially improving the approximation as the optimization progresses.

The cross-sectional area is minimized subject to two reliability constraints, which require the safety indices for strength and deflection constraints to be larger than three (probability of failure less than 0.00135). The RBDO problem, with the width w and thickness t of the beam as design variables that are assumed to be deterministic, can be formulated as

\[
\text{minimize } A = wt \\
\text{such that } \\
p - 0.00135 \leq 0 \tag{14}
\]

based on probability of failure, or

\[
\text{minimize } A = wt \\
\text{such that } \\
3 - \beta \leq 0 \tag{15}
\]

based on safety index, where \( \beta \) is the safety index, or

\[
\text{minimize } A = wt \\
\text{such that } \\
1 - P_f \leq 0 \tag{16}
\]

based on the probabilistic sufficiency factor.

The DRS has two design variables \( w \) and \( t \). A quadratic DRS in two variables has six coefficients to be estimated. Since Face Center Central Composite Design (FCCCD, e.g., Khuri and Cornell, 1996) is often used for quadratic RSA, a FCCCD with 9 points was employed here first with poor results. Based on our previous experience, higher order DRS are needed to fit the probability of failure or the safety index, and the number of points of a typical DOE should be about two times larger than the number of coefficients. A cubic RS with two variables has 10 coefficients that require about 20 design points. Latin Hypercube sampling (LHS) can be used as DOE for higher order RS (Qu et al., 2001). We found that LHS might fail to sample points near some corners of the design space, leading to poor accuracy around these corners. We therefore combined LHS with four vertices of the design space to obtain 20 points. Mixed stepwise regression (e.g., Myers and Montgomery, [15]) was employed to eliminate poorly characterized terms in the response surface models.

In order to demonstrate the utility of the \( P_f \) for estimating the required weight for correcting a safety deficiency, it is useful to see how the stresses and the displacements depend on the weight (or cross-sectional area) for this problem. If we have a given design with dimensions \( w_0 \) and \( t_0 \) and a \( P_f \) of \( P_{f0} \), which is smaller than one, we can make the structure safer by scaling both \( w \) and \( t \) uniformly by a constant \( c \)

\[
w = cw_0, \quad t = ct_0 \tag{17}
\]

It is easy to check from (12) (13) that the stress and the displacement will then change by a factor of \( c^3 \), and the area by a factor of \( c^2 \). Since the \( P_f \) is inversely proportional to the most critical stress or displacement, it is easy to obtain the relationship

\[
P_f = P_{f0}\left(\frac{A}{A_0}\right)^{1.5} \tag{18}
\]

where \( A_0 = w_0t_0 \). This indicates that a one percent increase in area (corresponding to 0.5 percent increase in \( w \) and \( t \)) will improve the \( P_f \) by about 1.5 percent. Since non-uniform increases in the width and thickness may be more efficient than uniform scaling, we may be able to do better than 1.5 percent. Thus, if we have \( P_{f0} = 0.97 \), we can expect that we can make the structure safe with a weight increase under two percent.

**DESIGN WITH STRENGTH CONSTRAINT**

The range for the DRS, shown in Table 2, was selected based on the mean-based deterministic design, \( w = 1.9574" \) and \( t = 3.9149" \). The probability of failure was calculated by direct MCS with 100,000 samples based on the exact stress in (12).

Cubic DRS with 10 coefficients were constructed and their statistics are shown in Table 3. An \( R^2_{adj} \) close to one and an average percentage error (defined as the ratio of RMSE predictor and mean of response) close to zero indicate good accuracy of the response surface. It is seen that the DRS for the PSF has the highest \( R^2_{adj} \) and the smallest average percentage error. The standard error in probability calculated by MCS can be estimated as

\[
\sigma_p = \sqrt{\frac{p(1-p)}{M}} \tag{19}
\]

where \( p \) is the probability of failure, \( M \) is the sample size of MCS. For 100,000 samples and a probability of failure of 0.2844 (the mean probability of failure in Table 3), the standard error due to the limited sampling is 0.00143. The RMSE error of the probability DRS is of 0.1103. Thus the error induced by the limited sampling (100,000) is much smaller than error of the response surface approximation to the probability of failure.

The probabilistic sufficiency factor DRS has an average error less than one percent, while the safety index DRS has an average error of about 15.6 percent. It must be noted, however, that the average percent errors of the three DRS cannot be directly compared, because one percent error in probabilistic sufficiency factor does not correspond to one percent error in probability of failure or safety index. Errors in safety index DRS were transformed to errors in terms of probability as shown in Table 3. It is seen that safety index DRS is more accurate than the probability DRS.

Besides average errors over the design space, it is instructive to compare errors measured in probability of failure in the important region of the design space. That is the region
containing the optimum. Here this region is the curve of target reliability according to each DRS, on which the reliability constraints is satisfied critically, and the probability of failure is predicted to be 0.00135. For each DRS, 11 test points were selected along a curve of target reliability as shown in the Appendix. The average percentage errors at these test points, shown in Table 4, demonstrate the probabilistic sufficiency factor DRS is much more accurate than the probability and safety index DRS. For the target reliability, the standard error due to MCS of 100,000 samples is 8.6%, which is comparable to the response surface error for the \( P_f \). For the other two response surfaces, the errors are apparently dominated by the modeling errors due to the cubic polynomial approximation. Since the major computational cost is calculating probability, the overall computational costs of constructing the three DRS are on similar level.

The optima found by using the DRS of Table 3 are compared in Table 5. The probabilistic sufficiency factor DRS clearly led to a better design, which has a safety index of 3.0 according to MCS. It is seen that the design from probabilistic sufficiency factor DRS is very close to the exact optimum. Note that the values of \( P_f \) for the probability based optimum and safety index based optimum provide a good estimate to the required weight increments. For example, with a \( P_f = 0.9663 \) the safety index based design has a safety factor shortfall of 3.37 percent, indicating that it should not require more than 2.25 percent weight increment to remedy the problem. Indeed the optimum design is 2.08 percent heavier. This would have been difficult to infer from a probability of failure of 0.00408, which is three times larger than the target probability of failure.

DESIGN WITH STRENGTH AND DISPLACEMENT CONSTRAINTS

For system reliability problem with strength and displacement constraints, the probability of failure is calculated by direct MCS with 100,000 samples based on the exact stress and exact displacement in (12) and (13). The allowable tip displacement \( D_0 \) is chosen to be 2.25" in order to have two competing constraints (Wu et al., 2001). The three cubic DRS in the range of design variables shown in Table 2 were constructed and their statistics are shown in Table 6.

The optima found by using the DRS of Table 6 are compared in Table 7. The probabilistic sufficiency factor DRS led to a better design than the probability or safety index DRS in terms of reliability. The probability of failure of current design is 0.00314 evaluated by MCS, which is higher than the target probability of failure of 0.00135. The deficiency of reliability in the current design is induced by the errors in the probabilistic sufficiency factor DRS. The design can be improved by performing another design iteration, which would reduce the errors in DRS by shrinking the design space around the current design.

However, with a \( P_f = 0.9733 \) we can tell that the deficiency in the PSF can be corrected by scaling up the area by a factor of 1.0182 according to (18). Since the area \( A \) is equal to \( c^2 w t \), the dimensions should be scaled by a factor \( c = 1.0091 \) to \( w = 2.7123 \) and \( t = 3.5315 \). Thus the objective function of the scaled design is 9.5785. The probability of failure of the scaled design is 0.001302 (safety index of 3.0110 and probabilistic sufficiency factor of 1.0011) evaluated by MCS with 1,000,000 samples.

RBDO USING DETERMINISTIC OPTIMIZATION WITH PROBABILISTIC SUFFICIENCY FACTOR

Wu et al. (1998, 2001) developed a safety-factor based approach for RBDO, where the reliability constraints are converted to equivalent deterministic constraints by using the concept of safety factor. The similarity between Wu's approach and the probabilistic sufficiency factor approach indicates that it may be worthwhile to study the use of probabilistic sufficiency factor in RBDO using deterministic optimization. The beam problem with strength reliability constraint was studied here. By starting from a mean value based design, where the deterministic safety factor is one, an initial design was found by deterministic optimization. Reliability analysis using Monte Carlo simulation shows the deficiency in probability of failure and probabilistic sufficiency factor. In the next design iteration, the safety factor of the next deterministic optimization is chosen to be

\[
\bar{s}(\mathbf{x}, \mathbf{d})^{(i+1)} \geq \frac{s(\mathbf{x}, \mathbf{d})^{(i)}}{P_f^{(i)}}
\]

which is used to reduce the yield strength of the material, \( R \). The optimization problem is formulated as

\[
\text{minimize } A = w t
\]

such that

\[
\sigma - \frac{R}{s(\mathbf{x}, \mathbf{d})^{(i+1)}} \leq 0
\]

The process is repeated until the optimum converges and the reliability constraint is satisfied. The design history is shown in Table 8. It is seen that the final design has a slightly higher weight than the optimum in Table 5. The reason is that this method employs a safety factor based on the mean values of the random variables, better prediction can be achieved by using an MPP based safety factor.

RBDO USING COARSE MCS WITH PROBABILISTIC SUFFICIENCY FACTOR

For many problems the required probability of failure is very low, so that good estimates require a very large MCS sample. In addition, the DRS must be extremely accurate in order to estimate well a very low probability of failure. Thus we may require an expensive MCS at a large number of design points in order to construct the DRS. The deterministic optimization described in the previous section may be used to solve this problem. However, since it does not use any derivative...
information for the probabilities, it is not likely to converge to an optimum design when competing failure modes are disparate in terms of the cost of improving their safety. A compromise between the deterministic optimization and the full probabilistic optimization is afforded by the \( P_{sf} \) by using an intermediate target probability \( P_f \) which is higher than the required probability \( P_r \) and can be estimated via a less expensive MCS and less accurate DRS. Then the \( P_{sf} \) can be recalibrated by a single expensive MCS. This is a variable-fidelity technique, with a large number of inexpensive MCS combined with a small number of expensive MCS.

For the beam example we illustrate the process by setting a low required probability of 0.0000135, and using as intermediate probability 0.00135, the value used as required probability for the previous examples. We start by finding an initial optimum design with the intermediate probability as the required probability. This involves the generation of a response surface approximation of \( P_{sf} \) for the intermediate probability as well as finding the optimum based on this response surface. We then perform an expensive MCS which is adequate for estimating the required probability. Here we use MCS with \( 10^7 \) samples. We now calculate the \( P_{sf} \) from this accurate MCS, and denote it \( P_{sf}^A \). At that design the \( P_{sf}^I \) predicted by the response surface approximation is about 1, because the initial optimization was performed with a lower limit of 1 on the \( P_{sf} \). In contrast, the accurate \( P_{sf}^A \) will, in general, be different for several reasons. These include the higher accuracy of the MCS, the response surface errors, and most important the lower probability requirements. For example, with \( 10^7 \) samples, at this initial design we may get \( P_{sf}^I = 1.01 \) for the intermediate probability (based on the 13500 lowest safety factors) and \( P_{sf}^A = 0.89 \) for the required probability (based on the 135 lowest safety factors).

With a value of \( P_{sf}^I \) and \( P_{sf}^A \) at the same point, we can define a scale factor \( f \), as the ratio of these two numbers

\[
f = \frac{P_{sf}^A}{P_{sf}^I}
\]

This ratio can be used to correct the response surface approximation during the optimization process. Once an optimum design is found with a given \( f \), a new accurate MCS can be calculated at the optimum, a new value of \( f \) can be calculated from Eq. (22) at the new point, and the process repeated until convergence. As further refinement, we have also updated the response surface for the intermediate probability, centering it about the new optimum.

The beam design with strength constraint was repeated here with a target probability of failure of 0.0000135, so that the previous response surface and the optimum in Table 5 is used here as initial design. The design process and ranges of DRS are shown in Table 9 and 10, respectively. It is seen that the target reliability is achieved in two design iterations of low fidelity DRS updated by two high fidelity reliability analysis.

CONCLUDING REMARKS

The paper presented a probabilistic sufficiency factor as a measure of the safety level relative to a target safety level, which can be obtained from the results of MCS with little extra computation. It was shown that a design response surface approximation can be more accurately fitted to the probabilistic sufficiency factor than to the probability of failure or the safety index. In the beam design example with single or multiple reliability constraints, it was demonstrated that the DRS based on probabilistic sufficiency factor has superior accuracy and accelerates the convergence of reliability-based design optimization. The probabilistic sufficiency factor also provides more information in regions of such low probability that the probability of failure or safety index cannot be estimated by MCS with a given sample size, which is helpful in guiding the optimizer. Finally it was shown that the probabilistic sufficiency factor can be employed by the designer to estimate the required additional weight to achieve a target safety level, which might be difficult with probability of failure or safety index.

In order to reduce the computational cost of RBDO, especially problems with very low probability of failure, two approaches based on the probabilistic sufficiency factor were proposed. A deterministic optimization based RBDO approach updates the safety factor based on the probabilistic sufficiency factor. A variable fidelity based RBDO approach can find satisfactory design with low probability of failure by using a correction factor to a response surface created by low-accuracy Monte Carlo Simulation.

ACKNOWLEDGEMENTS

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REFERENCE:


Table 1. Random variables

<table>
<thead>
<tr>
<th>Random variables</th>
<th>X</th>
<th>Y</th>
<th>R</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution</td>
<td>Normal (500, 100) lb</td>
<td>Normal (1000, 100) lb</td>
<td>Normal (40000,2000) psi</td>
<td>Normal (29E6, 1.45E6) psi</td>
</tr>
</tbody>
</table>

Table 2. Range of design variables for design response surface

<table>
<thead>
<tr>
<th>System variables</th>
<th>w</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>1.5&quot; to 3.0&quot;</td>
<td>3.5&quot; to 5.0&quot;</td>
</tr>
</tbody>
</table>

Table 3. Comparison of cubic design response surfaces (DRS) of probability of failure, safety index and probabilistic sufficiency factor for single strength failure mode (based on MCS of 100,000 samples)

<table>
<thead>
<tr>
<th>Error Statistics</th>
<th>LHS with 16 points + 4 vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability RS</td>
<td>Safety index RS</td>
</tr>
<tr>
<td>R²</td>
<td>0.9391</td>
</tr>
<tr>
<td>R²_adj</td>
<td>0.9228</td>
</tr>
<tr>
<td>RMSE Predictor</td>
<td>0.1103</td>
</tr>
<tr>
<td>Mean of Response</td>
<td>0.2844</td>
</tr>
<tr>
<td>APE (Average Percentage Error=RMSE Predictor/Mean of Response)</td>
<td>38.78%</td>
</tr>
<tr>
<td>APE in Pof (=RMSE Predictor of Pof/Mean of Pof)</td>
<td>38.78%</td>
</tr>
</tbody>
</table>

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Table 4. Errors in cubic DRS of probabilistic sufficiency factor, safety index and probability of failure at 11 points on the curve of target reliability

<table>
<thead>
<tr>
<th>DRS of</th>
<th>Probability of failure</th>
<th>Safety Index (Pof)</th>
<th>Probabilistic sufficiency factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Percentage Error in POF</td>
<td>213.86%</td>
<td>92.38%</td>
<td>10.32%</td>
</tr>
</tbody>
</table>

Table 5. Comparisons of optimum designs (strength only) based on cubic DRS of probabilistic sufficiency factor, safety index and probability of failure

<table>
<thead>
<tr>
<th>DRS of</th>
<th>Minimize objective function $F$ while $\beta \geq 3$ or $0.00135 \geq pof$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>Optima $w=2.6350, t=3.5000$</td>
</tr>
<tr>
<td>Safety index</td>
<td>$w=2.6645, t=3.5000$</td>
</tr>
<tr>
<td>Probabilistic sufficiency factor</td>
<td>$w=2.4526, t=3.8884$</td>
</tr>
<tr>
<td>Exact optimum (Wu et al., 2001)</td>
<td>$w=2.4484, t=3.8884$</td>
</tr>
</tbody>
</table>

Table 6. Comparison of cubic design response surfaces of probability of failure, safety index and probabilistic sufficiency factor for system reliability (strength and displacement)

<table>
<thead>
<tr>
<th>Error Statistics</th>
<th>Probability RS</th>
<th>Safety index RS</th>
<th>Probabilistic sufficiency factor RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>0.9392</td>
<td>0.9928</td>
<td>0.9997</td>
</tr>
<tr>
<td>$R^2_{adj}$</td>
<td>0.9231</td>
<td>0.9887</td>
<td>0.9996</td>
</tr>
<tr>
<td>RMSE Predictor</td>
<td>0.1234</td>
<td>0.3519</td>
<td>0.01055</td>
</tr>
<tr>
<td>Mean of Response</td>
<td>0.3839</td>
<td>1.3221</td>
<td>0.9221</td>
</tr>
<tr>
<td>APE (Average Percentage Error $=$ RMSE Predictor/ Mean of Response)</td>
<td>32.14%</td>
<td>26.62%</td>
<td>1.14%</td>
</tr>
<tr>
<td>APE in Pof ($=$ RMSE Predictor of Pof/ Mean of Pof)</td>
<td>32.14%</td>
<td>10.51%</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 7. Comparisons of optimum designs based on cubic DRS of probabilistic sufficiency factor, safety index and probability of failure (strength and displacement)

<table>
<thead>
<tr>
<th>DRS of</th>
<th>Minimize objective function $F$ while $\beta \geq 3$ or $0.00135 \geq pof$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>Optima $w=2.6591, t=3.5000$</td>
</tr>
<tr>
<td>Safety index</td>
<td>$w=2.6473, t=3.5000$</td>
</tr>
<tr>
<td>Probabilistic sufficiency factor</td>
<td>$w=2.6881, t=3.500$</td>
</tr>
</tbody>
</table>
Table 8. Design history of RBDO based on deterministic optimization with probabilistic sufficiency factor under strength constraint

<table>
<thead>
<tr>
<th>Probabilistic sufficiency factor</th>
<th>Optima</th>
<th>Objective function</th>
<th>Pof/Safety index/Sufficiency factor from MCS of $10^5$ samples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial design $(s_1^2=1.0)$</td>
<td>w=1.9574, t=3.9149</td>
<td>7.6630</td>
<td>0.49883/0.0029/0.7178</td>
</tr>
<tr>
<td>$s_2^2=s_1^2/0.7178$</td>
<td>w=2.1862, t=4.3724</td>
<td>9.5589</td>
<td>0.00140/2.9889/0.9986</td>
</tr>
<tr>
<td>$s_3^2=s_1^2/0.9986$</td>
<td>w=2.1872, t=4.3744</td>
<td>9.5676</td>
<td>0.00130/3.0115/1.0006</td>
</tr>
</tbody>
</table>

Table 9. RBDO using variable fidelity technique with probabilistic sufficiency factor under strength constraint

<table>
<thead>
<tr>
<th>Probabilistic sufficiency factor</th>
<th>Optima</th>
<th>Objective function</th>
<th>Pof/Safety index/Sufficiency factor from MCS of $10^7$ samples (Psf $^A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial design $(f_1=1.0)$</td>
<td>w=2.4526, t=3.8884</td>
<td>9.5367</td>
<td>0.001235/3.027/0.891938</td>
</tr>
<tr>
<td>$f_2=0.891938$</td>
<td>w=2.2522, t=4.6000</td>
<td>10.3600</td>
<td>0.000160/4.1587/0.996375</td>
</tr>
<tr>
<td>$f_3=0.996375$</td>
<td>w=2.4071, t=4.3000</td>
<td>10.3510</td>
<td>0.000108/4.2477/1.003632</td>
</tr>
</tbody>
</table>

Table 10. Range of design variables for design response surface

<table>
<thead>
<tr>
<th>System variables</th>
<th>w</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range for $f_2$ DRS</td>
<td>2.0&quot; to 3.0&quot;</td>
<td>3.6&quot; to 4.6&quot;</td>
</tr>
<tr>
<td>Range for $f_3$ DRS</td>
<td>1.7&quot; to 2.7&quot;</td>
<td>4.3&quot; to 5.3&quot;</td>
</tr>
</tbody>
</table>

APPENDIX: CONTOUR PLOT OF THREE DESIGN RESPONSE SURFACES AND TEST POINTS ALONG THE CURVE OF TARGET RELIABILITY

Figure A1. Contour plot of probabilistic sufficiency factor design response surface and test points along the curve of target reliability