Surrogate-based Aerodynamic Optimization under Uncertainty

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Abstract
Practical experiences reveal that aerodynamic design optima obtained from deterministic optimization may be sensitive to the manufacturing errors of aircraft shape and uncertain operating conditions. It is expected that aerodynamic optimization should consider the uncertainties. The major difficulty of aerodynamic optimization under uncertainties comes from the high computational cost of uncertainty analysis. This paper proposes an approach for aerodynamic optimization under uncertainties, in which the surrogate model is used to reduce the computational expense of uncertainty analysis. An airfoil optimization problem was used to test the approach. The results show that the optima obtained from this optimization formulation are less sensitive to the uncertainties, and the constraints are still satisfied under the uncertainties.

Keywords: Aerodynamic optimization; Uncertainty; Robust optimization; Surrogate model; airfoil

1 Introduction
Optimization methods have been widely used for aerodynamic design of aircraft. Traditionally, aerodynamics design optimization is formulated as a deterministic optimization problem at the given operating conditions. Unfortunately, practical experience suggests that aircraft does not frequently operate at the certain conditions. The operating conditions, such as lift coefficient, Mach number, Reynolds number may fluctuate during operation. In addition, there always exist small manufacturing errors for aircraft shape. Without considerations of those uncertainties, the aerodynamic design solution obtained by the deterministic optimization method may be highly sensitive to the variation of the operating conditions and the manufacturing errors of aircraft shape which will lead to performance (objectives) loss, or is unable to satisfy the given requirements (constraints). Therefore, an important issue in the aerodynamic optimization is the consideration of the uncertainties. It implies that the aerodynamic optimization solution should be made less sensitive to small changes in lift coefficient, Mach number, and Reynolds number, and to small manufacturing errors.

The aerodynamic optimization under uncertainties can be considered as a robust optimization problem in which a solution is sought that is relatively insensitive to small changes in the uncertain quantities. Robust optimization has been used in field of structural and mechanical design for many years. But the application of robust optimization to aerodynamic design is still quite new. The needs and opportunities for uncertainty-based multidisciplinary design methods for aerospace vehicles have been stated in the NASA report by Zang, et al [1].

One of the major barriers in the robust optimization of aerodynamic design is the computational expense of uncertainty analysis of a given design. The aerodynamic optimization under the variation of Mach number has been investigated using sensitivity derivatives method [2][3]. However, the computation of the sensitivity derivatives based on CFD simulations might be not accurate in some cases due to numerical noise in CFD simulation [4].

In this paper, we propose an approach for the aerodynamic optimization under uncertainties based on approximation model methods that are to construct a simplified mathematical approximation of the computationally expensive simulation and analysis code. The approximation models are then used in place of the original code to facilitate design optimization and the uncertainty analysis. Since the approximation model acts as a surrogate for the original code, it is often referred to as a surrogate model [5]. It is expected that the use of surrogate model is able to overcome the difficulty resulting from the computational expense of uncertainty analysis in the robust optimization of aerodynamic design.

In the next section, the aerodynamic optimization under uncertainties is formulated as a robust optimization problem.
in which the variations of the operating conditions and small manufacturing errors of the external shape are considered. In subsequent section, an uncertainty analysis approach, which is based on the surrogate model and Monte Carlo simulation, is presented. In section 4, an example of airfoil optimization under the variation of lift coefficient is used to illustrate and test the proposed approach. Several issues and the future work for further improvement are identified in the conclusions.

2 Formulation of aerodynamic optimization under uncertainty

In this section, we discuss the differences between the deterministic aerodynamic optimizations and the optimization under uncertainties using airfoil design as an example.

In the deterministic aerodynamic shape optimization of two-dimensional airfoil, the objective function drag coefficient \( C_d \) which relies on the shape design variables \( x \) and the operation condition parameter \( p \), is minimized over all possible designs, subject to constraints. The formulation is stated as.

\[
\text{Given : } \quad p \\
\text{Find : } \quad x \\
\text{Minimize : } \quad c_f \\
\text{Subject to : } \quad c_{ml} \leq c_m \leq c_{mu} \\
\quad V_l \leq V \\
\quad x_p \leq x \leq x_{io} \\
\]

where \( c_m \) is the pitch moment coefficient, \( V \) is the volume of airfoil, the subscript \( l \) and \( u \) represent the lower and upper specification limit, and the subscript \( k \) is the \( k \)-th design variable.

Since, in reality, there always exist variations of the operating conditions and small manufacturing errors for aircraft shape, design variables \( x \) and operating conditions \( p \) might not be the deterministic. It is necessary for the practical use of aerodynamic optimization to consider such uncertainties. The optimization under uncertainty is different from the deterministic aerodynamic optimizations in that the process of uncertainty analysis is needed for the optimization under uncertainties. The primary task of uncertainty analysis is to identify the probabilistic characteristics of response variables (model outputs) for a given uncertain inputs and their variability. The probabilistic characteristics include cumulative distribution function (cdf), probability density function (pdf), moments such as mean, standard deviation, skewness, and so on. Depending on specific applications, different probabilistic characteristics may be used. For example, for robust design, only the mean \( \mu \) and standard deviation \( \sigma \) may be used.

The airfoil optimization under uncertainty can be considered a robust design problem and formulated as following.

\[
\text{Given : } \quad \mathbf{p}_d, \mathbf{p}_p, \text{ and the random distribution function of } \mathbf{p}_p \text{ and } x_p \\
\text{Find : } \quad x_d, x_p \\
\text{Minimize : } \quad w_1 \frac{\mu_c}{s_1} + w_2 \frac{\sigma_c^2}{s_2} \\
\text{Subject to : } \quad c_{ml} + n\sigma_c \leq \mu_c \leq c_{mu} - n\sigma_c \\
\quad V_l + n\sigma_V \leq \mu_v \\
\quad x_{dl} \leq x \leq x_{du}, \quad x_{pl} + n\sigma_{x_p} \leq \mu_{x_p} \leq x_{pu} - n\sigma_{x_p} \\
\]

Where design variables \( x_d \) are deterministic variables and the \( x_p \) are probabilistic variables, \( i \) is the \( i \)-th uncertain design variable, and the vector of design parameters consists of the deterministic the subset vectors of deterministic parameter \( \mathbf{p}_d \) and the probabilistic parameter \( \mathbf{p}_p \), \( w_1 \) and \( w_2 \) are the weights, and \( s_1 \) and \( s_2 \) are the scale factors for the objective components. The constraints are formulated to include a number of standard deviations \( n \) within specification limits. The number is a measure of quality in the robust optimization.

Compared to the deterministic aerodynamic optimizations, the optimization of aerodynamic design under uncertainty requires an additional process, i.e. uncertainty analysis which increases the computational expense. To mitigate the burden of computation expense, it is essential to use the efficient method for uncertainty analysis in the aerodynamic optimization under uncertainty.
3 The methods for uncertainty analysis

As mentioned above, the uncertainty analysis is a major task in the optimization under uncertainty. The objective of this step is to characterize the uncertainties of the output (aerodynamic performance) given the input uncertainties. Uncertainty in design inputs can be characterized in many ways. It can be specified by various types of probability density functions, such as normal, log-normal, uniform, and so on. More detailed discussions of the general mathematical framework for characterization of uncertainties can be found in [6].

Given the definition of uncertain inputs and their variability, the resulting variability of outputs can be measured. Multiple techniques for assessing performance uncertainty exist; three are popular recently: sensitivity-based variability estimation, design of experiments, and Monte Carlo simulation [7].

1) Sensitivity-Based Variability Estimation: These methods are based on Taylor’s series expansions. Many researches have utilized either first order or second order approximation, neglecting the higher order terms, overcoming the time consuming calculations. This approach is recommended when responses are known to be close to linear, and also when rough estimates are acceptable.

2) Design of Experiments: A second approach is through more structured sampling, using a designed experiment, a common tool for robust design analysis. This approach is recommended if the inputs are defined with interval bounds or membership functions. The computational cost depends on the particular experiment design chosen and the levels of the uncertain parameter.

3) Monte Carlo Simulation: The concept of this method is straightforward, which is implemented by randomly simulating a population of designs, given the stochastic properties of one or more random variables, and generates the probability space of the performance responses through integration. Simple random sampling and description sampling are two techniques often used for Monte Carlo simulation, and the latter is a more efficient variance reduction strategy. Among those methods, the Monte Carlo simulation is the most accurate method and is not limited to specific problems. However, Monte Carlo method requires a large number of performance evaluations which makes it impractical for many design problems. Recently, the uncertainty analysis strategy based on surrogate models has been suggested by several research groups [7-10]. In this work, the original detailed analysis model (CFD simulation) for performance evaluations is replaced with a surrogate model during optimization to reduce the computational expense significantly. Various Surrogate model are available to be utilized, such as response surface models (RSM), radial basis functions (RBF), kriging models, and so on. Because the aerodynamic problems may be often high non-linear, the surrogate model chosen in this paper is kriging model, which is accurate enough to replace complex analysis codes. The reference [11] provides the details on kriging modeling.

After the surrogate model is generated, additional sample points are used to verify the accuracy of the kriging model. In this study, the maximum absolute error and R Square are chosen to measure the surrogate model. They are defined as

\[ MAXERR = \text{MAX}[|y_i - \bar{y}_i|, i = 1,2,...,n]\]

\[ R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \bar{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \]

Where \( \bar{y}_i \) is the mean of \( i \)-th response variable \( y_i \) (performance), \( n_s \) is the number of the test points, which is recommended to choose \( 0.1N \) or \( \sqrt{N} \) [12], and \( N \) represents the number of sample points used to construct the kriging model.

4 The procedure of aerodynamic optimization under uncertainty

Using the uncertainty analysis method presented in the previous section, this section provides a step-by-step description of the procedure for aerodynamic optimization under uncertainty. The flowchart of the procedure is illustrated in Fig.1.

Step 1: Select a proper design of experiment (DOE) method for generating the \( n \) sample points in the design space of \((x_{pa}, x_{pe}, p, p_f)\), then use CFD codes to evaluate the performance responses of the sample points.

Step 2: Construct a kriging model using the data of the sample points. This kriging model is an approximation for the performance predictions by the CFD codes.

Step 3: Conduct and operate the deterministic optimization based on the kriging model to obtain the optimum of
deterministic optimization. The mean and variance of this point will be used to replace $s_1$ and $s_2$ defined in Eq.2, when formulating the robust optimization.

Step 4: Formulate the robust design optimization problem. The parameters, design variable, objective and the constraints are formulated as presented in Eq. 2.

Step 5: A global optimization algorithm is adopted to solve the robust optimization formulation.

Step 6: If the convergence criteria are satisfied, go to Step 7, and the process terminates. Otherwise, go back to Step 5.

4 Test example

An airfoil optimization under the variation of lift coefficient $C_l$ is used as an example to test the approach above. The distribution of lift coefficient is assumed to be a normal distribution with mean value $\mu_{C_l}=0.45$ and standard deviation $\sigma_{C_l}=0.045$. The objective is to minimize the drag coefficient $C_d$ and its variance $\sigma_{\beta_d}^2$ under the variation of lift coefficient $C_l$. In this work, a modified “PARSEC” method [13,14] is used to describe the airfoil shape with twelve design variables ($R_{lep}$, $X_{up}$, $Z_{up}$, $Z_{xxup}$, $R_{lelo}$, $X_{lo}$, $Z_{lo}$, $Z_{xxlo}$, $\alpha_{te}$, $\beta_{te}$, $Z_{te}$, $\Delta Z_{te}$) as shown in Fig.2, and $Z_{te}$ and $\Delta Z_{te}$ are fixed to zero for simplification, i.e., the design variable $x$ is a vector of ten dimensions. Constraints are imposed on the pitching moment coefficient $C_m$ and the thickness of airfoil $t$. Table 1 depicts the starting point and the range limits of design variables and constraints.

**Table 1. Design variables and constraints**

<table>
<thead>
<tr>
<th>Variables</th>
<th>$R_{lep}$</th>
<th>$X_{up}$</th>
<th>$Z_{up}$</th>
<th>$Z_{xxup}$</th>
<th>$R_{lelo}$</th>
<th>$X_{lo}$</th>
<th>$Z_{lo}$</th>
<th>$Z_{xxlo}$</th>
<th>$\alpha_{te}$</th>
<th>$\beta_{te}$</th>
<th>$C_d$</th>
<th>$C_m$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower</td>
<td>0.025</td>
<td>0.3</td>
<td>0.11</td>
<td>-0.9</td>
<td>0.019</td>
<td>0.15</td>
<td>-0.045</td>
<td>0.4</td>
<td>2</td>
<td>20</td>
<td>---</td>
<td>-1.0</td>
<td>0.15</td>
</tr>
<tr>
<td>Baseline</td>
<td>0.027</td>
<td>0.346</td>
<td>0.114</td>
<td>-0.86</td>
<td>0.020</td>
<td>0.16</td>
<td>-0.0424</td>
<td>0.435</td>
<td>4.5</td>
<td>25</td>
<td>0.00641</td>
<td>-0.867</td>
<td>0.1527</td>
</tr>
<tr>
<td>Upper</td>
<td>0.030</td>
<td>0.5</td>
<td>0.12</td>
<td>-0.8</td>
<td>0.021</td>
<td>0.25</td>
<td>-0.04</td>
<td>0.46</td>
<td>8</td>
<td>30</td>
<td>0.00641</td>
<td>0</td>
<td>---</td>
</tr>
</tbody>
</table>

Figure 2. “PARSEC” airfoil geometry defined by 12 variables

**Table 2. Accuracy of the kriging model**

<table>
<thead>
<tr>
<th></th>
<th>$C_d$</th>
<th>$C_m$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAXERR</td>
<td>0.000426</td>
<td>0.00593</td>
<td>0.000636</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.950464</td>
<td>0.99386</td>
<td>0.997772</td>
</tr>
</tbody>
</table>
Now we follow the procedure presented in section 3 to solve this aerodynamic optimization problem for the airfoil.

Step 1: The ten design variables and the uncertain parameter $c_i$ are used as inputs to construct surrogate model for $c_d$, $c_m$ and $t$. The 150 sample points are chosen using Latin Hypercube design where the points are so selected that they are uniformly distributed for each dimension. Given the design variables of the airfoil and the operation conditions (lift coefficient, $\mu_{cl}=0.45$, $\sigma_{cl}=0.045$; Mach number, $M=0.4$; Reynolds number, $Re=5\times10^6$), an airfoil analysis code Xfoil[15] is used to predict the airfoil performance, including drag coefficient $c_d$, the pitching moment coefficient $c_m$ and the thickness of airfoil $t$.

Step 2: Construct the kriging model and test the model utilizing other 13 points generated with Latin Hypercube design. Tab 2 shows the accuracy of the kriging model for three performance responses. The accuracy of the kriging model is reasonably satisfied.

Step 3: The deterministic optimization was conducted using GA based on kriging model, and the results are listed in Tab.3 where $R(c_m)$ in the last row represents the probability for satisfaction of $c_m$ constraint.

Step 4: The formulation of the robust optimization is defined as

$$
\begin{align*}
given & \quad \Re, M, \mu_{cl}, \sigma_{cl} \\
find & \quad R_{leup}, X_{up}, Z_{up}, Zx_{up}, R_{lelo}, X_{lo}, Z_{lo}, Zx_{lo}, \alpha te, \beta te \\
minimize & \quad f = w_1 \mu_{cl} + w_2 \frac{\sigma_{cl}^2}{\sigma_{c_d}^2} \\
subject to & \quad c_m + 3\sigma_{c_m} \leq \mu_{c_m} \leq c_m - 3\sigma_{c_m} \\
& \quad \mu_{c_d} \leq c_{d0} ; \quad t_1 \leq t ; \quad x_d \leq x_i \leq x_u
\end{align*}
$$

Where $\mu_{c_d}$ and $\sigma_{c_d}^2$ are the mean and variance of the drag coefficient for the optimum airfoil obtained from the deterministic optimization, $c_{d0}$ is the drag coefficient of the starting point. The $w_1$ and $w_2$ in this example are taken as 1 and 4 respectively. The reason is that it’s expected that the mean of $c_d$ should be closed to $\mu_{c_d}$ obtained by the deterministic optimization, and its standard deviation $\sigma_{c_d}$ should reach a half of $\sigma_{c_d}^*$. Uncertain constraints are only imposed on $c_m$, while the thickness $t$ is a deterministic constraint.

Step 5: The optimization problem under uncertainty is solved with Monte Carlo simulation based on kriging model. The genetic algorithm (GA) is chosen to find the optimization in this example. The results from the robust optimal are shown in Tab.3.

### Table 3. Optima of the two optimizations

<table>
<thead>
<tr>
<th>Design variables, objective and constrants</th>
<th>Deterministic optimization</th>
<th>Robust optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{leup}$</td>
<td>0.026036</td>
<td>0.025378</td>
</tr>
<tr>
<td>$X_{up}$</td>
<td>0.449981</td>
<td>0.388554</td>
</tr>
<tr>
<td>$Z_{up}$</td>
<td>0.115881</td>
<td>0.113023</td>
</tr>
<tr>
<td>$Zx_{up}$</td>
<td>-0.882433</td>
<td>-0.816693</td>
</tr>
<tr>
<td>$R_{lelo}$</td>
<td>0.019610</td>
<td>0.019957</td>
</tr>
<tr>
<td>$X_{lo}$</td>
<td>0.175463</td>
<td>0.150056</td>
</tr>
<tr>
<td>$Z_{lo}$</td>
<td>-0.044773</td>
<td>-0.042501</td>
</tr>
<tr>
<td>$Zx_{lo}$</td>
<td>0.435150</td>
<td>0.459426</td>
</tr>
<tr>
<td>$\alpha te$</td>
<td>4.150998</td>
<td>7.037036</td>
</tr>
<tr>
<td>$\beta te$</td>
<td>29.287078</td>
<td>22.3158</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean value of performances</th>
<th>$\mu_{c_d}$</th>
<th>$\mu_{c_m}$</th>
<th>$\mu_{cl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.153682</td>
<td>-0.094273</td>
<td>0.155392</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Standard deviation of performances</th>
<th>$\sigma_{c_d}$</th>
<th>$\sigma_{c_m}$</th>
<th>$\sigma_{cl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.003193</td>
<td>0.003017</td>
<td>3.6995e-5</td>
</tr>
</tbody>
</table>

| $R(c_m)$ | 0.963575 | 1 |

The comparison of the results from these two optimizations indicates that although $c_d$ obtained from the robust optimization is relatively larger than that obtained from deterministic optimization, $\sigma_{c_d}$ becomes much smaller and
$R(C_m)$ increases after robust optimization. It means that the drag coefficient $c_d$ is less sensitive to the uncertain parameter $c_l$ and the constraint for pitch moment $c_m$ is still satisfied under the uncertainties.

**4 Conclusions**

In this work, the aerodynamic optimization problem under uncertainty was formulated. The Monte Carlo simulation was selected for uncertainty analysis to the optimization problem. To reduce the expensive computational cost, an approach, that uses Monte Carlo simulation based on surrogate model to assess the uncertainty of performance responses, was proposed to solve the aerodynamic optimization problem under uncertainty. In this approach, the kriging model was used as a surrogate model for CFD simulation. An airfoil optimization problem under the variation of lift coefficient was selected to test the approach. The preliminary results from the test example are quite encouraging. The surrogate-based uncertainty analysis method is able to significantly reduce computational expense in finding a robust optimization solution in which the object (drag) is less sensitive to the uncertain quantities and the constraint (pitch moment) is still satisfied under the uncertainties. The results indicated this method is feasible and efficient for the airfoil optimization under uncertainties. However, the accuracy of surrogate models is very important during the process. If the surrogate model is not accurate enough, the accuracy of variance of performances predicted based on surrogate model may get worse and the results from optimization under uncertainty might be unreliable. Future work will develop more efficient formulation of optimization under uncertainty, and improve the uncertainty analysis method and the approximation strategies.

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