ABSTRACT
Probabilistic design to address uncertainty and variability has been approached from many different angles, by different communities. While reliability methods focus on probability of constraint satisfaction or violation, robust design methods have focused primarily on the level of performance variation, the sensitivity of design objectives. “Six Sigma” quality concepts have arisen more recently from the manufacturing arena, focusing on measuring and controlling variation. All of these approaches deal with some aspect of modeling uncertainty or variability and measuring, improving, or controlling performance variation. For the case of six sigma, the term “Design for Six Sigma (DFSS)” has been coined and is the current push in industry; however, its implementation often does not involve design.

In an engineering design context, the concepts and philosophy of Six Sigma can be combined with methods from structural reliability and robust design to formulate a strategy to “optimize for six sigma” quality. Such a strategy is presented in this paper, a six sigma based probabilistic design optimization formulation. Through combining these concepts and approaches, variability is incorporated within all elements of this probabilistic optimization formulation – input design variable bound formulation, output constraint formulation, and robust objective formulation. This six sigma based probabilistic design optimization formulation, as implemented within the iSIGHT design framework, is demonstrated in this paper for the structural design of a welded joint. Results presented illustrate the trade off between performance and quality when optimizing for six sigma quality.

1 INTRODUCTION
Optimization without including uncertainty leads to designs that cannot be called “optimal”, but instead are potentially high risk solutions that likely have a high probability of failing in use. Optimization algorithms tend to push a design towards one or more constraints until the constraints are active, leaving the designer with a design for which even slight uncertainties in the problem formulation or changes in the operating environment could produce failed designs. While most optimization problem formulations and solution strategies are deterministic, very few real engineering problems are void of uncertainty; variation is inherent in material characteristics, loading conditions, simulation model accuracy, geometric properties, manufacturing precision, actual product usage, etc. Traditionally, many uncertainties are removed through assumptions, and others are handled through crude safety factors methods, which often lead to over-designed products and do not offer insight into the effects of individual uncertainties and the actual margin of safety of a design.

Probabilistic methods have been developed to convert deterministic problem formulations into probabilistic formulations to model and assess the effects of known uncertainties. Until very recently, however, the computational expense of probabilistic analysis of a single design point has made its application to all but very simplistic design problems impractical.1 Consequently, probabilistic optimization has been considered prohibitively expensive, particularly for complex multidisciplinary problems. With the steady increases in computing power, large scale parallel processing capabilities, and availability of probabilistic analysis and optimization tools and systems, however, the combination of these technologies can facilitate effective probabilistic analysis and optimization for complex design problems.

The probabilistic/statistical design methods developed in recent decades have come from several different communities: structural reliability1-9, Taguchi quality engineering10-12, and more recently “six sigma” quality engineering13-15. Each of these classes of approaches has a specific focus in probabilistic analysis and/or optimization. Structural reliability analysis is geared towards assessing the probability of failure of a design with respect to specified structural and/or performance constraints and the evaluated variation of these constraint functions. Thus a deterministic design problem is converted to reliability analysis problem by converting deterministic constraints into probabilistic constraints.
Thus the focus here is again on deviations) performance variation within design with a focus on maintaining analysis methods used to capture performance variation, of these probabilistic constraints.\textsuperscript{1,16-18} The heart of DFSS methods is DOE and other statistical probability distributions and properties, on satisfaction of these probabilistic constraints.\textsuperscript{1,16-18}

With Taguchi based quality engineering methods, the focus is on performance objectives, which are expanded from deterministic “minimize” or “maximize” objectives to include both mean performance and performance variation. With Taguchi methods, the goals are to drive mean performance towards a target, “mean on target”, and to “minimize variance” of performance.\textsuperscript{12} Taguchi methods employ metrics, such as Signal-to-Noise ratio and Loss function, to achieve these goals.\textsuperscript{10,11} However, within Taguchi based methods, constraints are not formulated as typically done with optimization formulations. Taguchi methods employ design of experiments (DOE)\textsuperscript{19} to evaluate potential designs. The best alternative, with respect to the chosen objective metrics, is selected from among those evaluated. Optimization is not generally performed between evaluated design points.

The rapidly growing current push in industry, with respect to managing uncertainty and seeking “quality” products, is Design for Six Sigma or DFSS. At the heart of DFSS methods is DOE and other statistical analysis methods used to capture performance variation, with a focus on maintaining $\pm 6\sigma$ (±6 standard deviations) performance variation within design specification limits.\textsuperscript{1,3-12} Thus the focus here is again on constraints, as with the probabilistic constraints in reliability-based design. With DFSS arising from manufacturing quality control pushes, DOE with a minimum number of physical experiments is often critical. Optimization is again generally not performed, as with Taguchi methods, since physical experiments are conducted; optimization is usually not even mentioned in the few available DFSS references. In this paper a six sigma based probabilistic design optimization formulation is presented that combines approaches from structural reliability and robust design with the concepts and philosophy of six sigma to facilitate comprehensive probabilistic optimization. Variability is incorporated within all elements of this probabilistic optimization formulation – input design variable bound formulation, output constraint formulation, and robust objective formulation. An overview of six sigma concepts is provided in the next section to establish context. The six sigma based probabilistic design optimization formulation is then presented in Section 3. Implemented within the iSIGHT design environment, this formulation is demonstrated in Section 4 for the structural design of a welded joint. A summary and discussion of complementary facilitating technologies is provided in closing in Section 5.

### 2 SIX SIGMA RELIABILITY AND ROBUSTNESS

In order to improve design quality, or optimize for design quality, design quality must first be measured. Two measures of design quality, within two separate communities as mentioned in the previous section, are reliability and robustness. One drawback of reliability-based optimization methods is that the objective is evaluated at the mean value point. The focus is on constraints, shifting response distributions away from constraint boundaries, but not on the size of the response distributions and the possibility of reducing response variation. Robust design, however, is generally focused on reducing response variation, balancing “mean on target” and “minimize variation” performance objectives. The term “robustness” in the robust engineering design context is defined as the sensitivity of performance parameters to fluctuations in design parameters, particularly uncertain design parameters. This sensitivity is captured through performance variability estimation.
The one consistency between the measure of reliability and the measure of robustness is the measure of variation: sigma. The term “sigma” refers to standard deviation, \( \sigma \). Standard deviation or variance, \( \sigma^2 \), is a measure of dispersion of a set of data around the mean value (\( \mu \)) of this data. This property can be used both to describe the known variability of factors that influence a system (product or process), and as a measure of performance variability, and thus reliability, robustness, or simply quality. Performance variation can be characterized as a number of standard deviations from the mean performance, as shown in Figure 1. The areas under the normal distribution in Figure 1 associated with each \( \sigma \)-level relate directly to the probability of performance falling in that particular range (for example, \( \pm 1\sigma \) is equivalent to a probability of 0.683). These probabilities are displayed in Table 1 as percent variation (equivalent to reliabilities) and number of defective parts per million parts.

![Figure 1 Normal Distribution, 3-\( \sigma \) Design](image)

**Table 1  Sigma Level as Percent Variation and Defects per Million**

<table>
<thead>
<tr>
<th>Sigma Level</th>
<th>Percent Variation</th>
<th>Defects/million (short term)</th>
<th>Defects/million (long term)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 1\sigma )</td>
<td>68.26</td>
<td>317,400</td>
<td>697,700</td>
</tr>
<tr>
<td>( \pm 2\sigma )</td>
<td>95.46</td>
<td>45,400</td>
<td>308,733</td>
</tr>
<tr>
<td>( \pm 3\sigma )</td>
<td>99.73</td>
<td>2,700</td>
<td>66,803</td>
</tr>
<tr>
<td>( \pm 4\sigma )</td>
<td>99.9937</td>
<td>63</td>
<td>6,200</td>
</tr>
<tr>
<td>( \pm 5\sigma )</td>
<td>99.999943</td>
<td>0.57</td>
<td>233</td>
</tr>
<tr>
<td>( \pm 6\sigma )</td>
<td>99.9999998</td>
<td>0.002</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Quality can be measured using any of the variability metrics in Table 1—“sigma level”, percent variation or probability/reliability, or number of defects per million parts—by comparing the associated performance specification limits and the measured performance variation. In Figure 1, the lower and upper specification limits that define the desired performance range are shown to coincide with \( \pm 3\sigma \) from the mean. The design associated with this level of performance variance would be considered a “3\( \sigma \)” design. Is this design of acceptable quality? Traditionally, if \( \pm 3\sigma \) worth of performance variation was identified to lie within the set specification limits this was viewed as acceptable variation; in this case, 99.73% of the variation is within specification limits, or the probability of meeting the requirements defined by these limits is 99.73%. In engineering terms, this probability was deemed acceptable.

More recently, however, the 3\( \sigma \) quality level has been viewed as insufficient quality, initially from a manufacturing perspective, and then extended into an engineering design perspective. Motorola, in defining “six sigma quality”, translated the sigma quality level to the number of defective parts per million (ppm) parts being manufactured. In this case, as can be seen in Table 1, \( \pm 3\sigma \) corresponds to 2700 ppm defective. This number was deemed unacceptable. Furthermore, Motorola and others observed that even at some observed variation level, mean performance could not be maintained over time. If a part is to be manufactured to some nominal specification, plus/minus some specification limits, the mean performance will change and thus the distribution will shift. A good example of this is tool wear. If a process is set up to manufacture a part to a nominal dimension of 10 in. with a tolerance of \( \pm 0.10 \) in., even if the process meets this nominal dimension on average, the cutting tool will wear with time, and the average part dimension will shift, say to 10.05 in. This will cause the distribution of performance variation to shift, while the specification limits remain fixed, and thus the area of the distribution outside one of the specification limits will increase.

This shift was observed by Motorola and others to be approximately 1.5\( \sigma \), and was used to define “long term sigma quality” as opposed to “short term sigma quality”. This explains the last column in Table 1. While the defects per million for short term correspond directly to the percent variation for a given sigma level associated with the standard normal distribution, the defects per million for long term correspond to a 1.5\( \sigma \) shift in the mean. In this case, 3\( \sigma \) quality leads to 66,803 defects per million, which is certainly undesirable. Consequently, Motorola defined a quality goal of \( \pm 6\sigma \) and “six sigma quality” came to define the desired level of acceptable performance variation. With this quality goal, the defects per million, as shown in Table 1, is 0.002 for short term quality, and 3.4 for long term quality; both acceptable quality levels.

The focus on achieving six sigma quality is commonly referred to as Design for Six Sigma (DFSS). In a probabilistic engineering design context, DFSS can be implemented by incorporating the “minimize performance variation” goal of robust design, qualified
by the constraint of reliability, by striving to maintain six-sigma (±6σ) performance variation within the defined acceptable limits, as illustrated in Figure 2. In Figures 2a and 2b, the mean, µ, and lower specification limit (LSL) and upper specification limit (USL) on performance variation are held fixed. Figure 2a represents the 3σ design of Figure 1; ±3σ worth of performance variation is within the defined specification limits. In order to achieve a 6σ design, one for which the probability that the performance will remain within the set limits is essentially 100%, the performance variation must be reduced (reduced σ_y), as shown in Figure 2b.

![3-σ Design](image1.png) ![6-σ Design](image2.png)

**Figure 2** Design for Six Sigma

### 3 IMPLEMENTATION: SIX SIGMA BASED PROBABILISTIC DESIGN

Combining the probabilistic, uncertainty elements of reliability, robust design, and six sigma, a six sigma based probabilistic design optimization approach is presented here that incorporates variability through definition of random variables and their distributions and statistical properties, input constraints (bounds of random design variables), output constraints (sigma level quality constraints or reliability constraints), and objective robustness (minimize variation). This approach is implemented within the ISIGHT design environment. The six sigma based probabilistic design optimization formulation is described first in Section 3.1. The key to implementing this probabilistic optimization formulation, to optimize for six sigma quality, is the ability to estimate response mean and standard deviation or variance during optimization. Three classes of methods for doing so are discussed in Section 3.2.

#### 3.1 Six Sigma Probabilistic Design Optimization Formulation

The six sigma based probabilistic design optimization formulation is given as follows:

- **Find** the set of design variables X that:
- **Minimizes:** \( F(\mu_y(X), \sigma_y(X)) \)
- **Subject to:**
  - \( g(\mu_y(X), \sigma_y(X)) \leq 0 \) [3]
  - \( X_L + n\sigma_x \leq \mu_x \leq X_U - n\sigma_x \)

Here X includes input parameters which may be design variables, random variables, or both. Both input and output constraints are formulated to include mean performance and a desired “sigma level”, or number of standard deviations within specification limits:

- \( \mu_y - n\sigma_y \geq \text{Lower Specification Limit} \) [4]
- \( \mu_y + n\sigma_y \leq \text{Upper Specification Limit} \) [5]

The robust design objective for this formulation, including “mean on target” and “minimize variation” robust design goals, is generally formulated as follows:

\[
F = \sum_{i=1}^{l} \left[ \frac{w_1}{s_1}(\mu_i - M_i)^2 + \frac{w_2}{s_2}\sigma_i^2 \right] \tag{6}
\]

where \( w_1 \) and \( w_2 \) are the weights and \( s_1 \) and \( s_2 \) are the scale factors for the “mean on target” and “minimize variation” objective components respectively for performance response \( i \), \( M_i \) is the target for performance response \( i \), and \( l \) is the number of performance responses included in the objective. For the case in which the mean performance is to be minimized or maximized, rather than directed towards a target, the objective formulation of Eqn. 6 can be modified as shown in Eqn. 7, where the first term is positive when the response mean is to be minimized, and negative when the response mean is to be maximized.

\[
F = \sum_{i=1}^{l} \left[ (+/-) \frac{w_1}{s_1}\mu_i + \frac{w_2}{s_2}\sigma_i^2 \right] \tag{7}
\]

#### 3.2 Measuring Variability: Sampling Methods

To implement the six sigma based probabilistic optimization formulation presented in the previous section, to calculate the robust objective value and quality with respect to constraints (sigma level or reliability), the mean and standard deviation or variance must be estimated for all outputs during optimization. The methods and tools developed for structural reliability analysis and robust design facilitate this estimation. Three classes of methods that have been used with this formulation are discussed in this section:

1. Monte Carlo Simulation
   - Simple Random Sampling \(^{21}\)
   - Descriptive Sampling \(^{19,23}\)
2. Design of Experiments \(^{19}\)
3. Sensitivity Based Estimation
   - First Order Taylor’s Expansion \(^{12,24}\)
   - Second Order Taylor’s Expansion \(^{25}\)

\(^{1}\) NOTE: Limitations are known to exist with weighted sum type objective functions when generating Pareto sets \(^{20}\); development of alternate formulations is a topic of current research.
Monte Carlo Simulation (MCS)

Monte Carlo simulation techniques are implemented by randomly simulating a design or process, given the stochastic properties of one or more random variables, with a focus on characterizing the statistical nature (mean, variance, range, distribution type, etc.) of the responses (outputs) of interest.²¹ Monte Carlo methods have long been recognized as the most exact method for all calculations that require knowledge of the probability distribution of responses of uncertain systems to uncertain inputs. To implement a Monte Carlo simulation, a defined number of system simulations to be analyzed are generated by sampling values of random variables (uncertain inputs), following the probabilistic distributions and associated properties defined for each.

Several sampling techniques exist for simulating a population; two techniques are discussed here: simple random sampling,²¹ the tradition approach to Monte Carlo, and descriptive sampling,²²,²³ a more efficient variance reduction technique. These two techniques are compared in Figure 3. With simple random sampling, the traditional MCS approach shown in Figure 3a, sample points are generated randomly from each distribution, as its name implies. Sufficient sample points (often tens of thousands) must be taken to ensure the probability distributions are fully sampled.

![Figure 3 Monte Carlo Sampling Comparison](image)

(a) Simple Random  (b) Descriptive

Variance reduction sampling techniques have been developed to reduce the sample size (number of simulations) without sacrificing the quality of the statistical description of the behavior of the system. Descriptive sampling is one such sampling technique. In this technique, the probability distribution of each random variable is divided into subsets of equal probability and the analysis is performed with each subset of each random variable only once (each subset of one random variable is combined with only one subset of each other random variable). This sampling technique is similar to Latin Hypercube experimental design techniques, and is best described through illustration as in Figure 3b for two random variables. Each row and column in the discretized two variable space is sampled only once, in random order. Only seven points/subsets are shown in Figure 3b for clarity in illustration; obviously more points are necessary for acceptable estimation, but often an order of magnitude fewer sample points are necessary for statistical estimates when compared to simple random sampling.

Design of Experiments (DOE)

A second approach for estimating performance variability due to uncertain design parameters is through more structured sampling, using a designed experiment, a common tool for robust design analysis. In DOE, a design matrix is constructed, in a systematic fashion, that specifies the values for the design parameters (uncertain parameters in this context) for each sampled point, or experiment. A number of experimental design techniques exist for efficiently sampling values of design parameters; for more details on experimental design techniques, see Ref. 19.

With DOE, potential values for uncertain design parameters are not defined through probability distributions, but rather are defined by a range (low and high values), a nominal baseline plus/minus some delta or percent, or through specified values, or levels. In this case each run in the designed experiment is a combination of the defined levels of each parameter. Consequently, this approach represents a higher level of uncertainty in uncertain design parameters, with expected ranges rather than specific distributions.

The computational cost of implementing DOE depends on the particular experiment design chosen and the levels of the uncertain parameters, but is generally significantly less than that of Monte Carlo simulation. However, the estimates may not be as accurate. This method is recommended when distributions are not known, and can also be used when uncertain parameters are known to vary uniformly.

Sensitivity Based Variability Estimation

A third approach for estimating performance variability is a sensitivity-based approach, based on Taylor’s series expansions. In this approach, rather than sampling across known distributions or ranges for uncertain design parameters, gradients for performance parameters are taken with respect to the uncertain design parameters, and hence the “sensitivity-based” nature of the approach. Generally with this approach the Taylor’s expansion is either first order, neglecting higher order terms, or second order. Obviously there is a tradeoff between expense (number of gradient calculations) and accuracy when choosing to include or neglect the higher order terms. The first order and second order formulations are given as follows.

*First Order Taylor’s Expansion.* Neglecting higher order terms, the Taylor’s series expansion for a performance response, Y, is:
The mean of this performance response is then calculated by setting the uncertain design parameters to their mean value, $\mu_i$:

$$\mu_i = Y(\mu_i)$$  \[9\]

and the standard deviation of $Y(x)$ is given by:

$$\sigma_i = \left( \sum_{i=1}^{n} \left( \frac{\partial Y}{\partial x_i} \right)^2 \sigma_{x_i}^2 \right)^{1/2}$$  \[10\]

where $\sigma_{x_i}$ is the standard deviation of the $i$th parameter and $n$ is the number of uncertain parameters. For more details on this approach for variability estimation for robust design purposes, see Refs. 12 and 24.

Since first order derivatives of responses with respect to random variables are needed in Eqn. 10, and the mean value point is needed in Eqn. 9, the first order Taylor’s expansion estimates require $n+1$ analyses for evaluation. Consequently, this approach is significantly more efficient than Monte Carlo simulation, and often more efficient than DOE while including distribution properties. However, the approach loses accuracy when responses are not close to linear in the region being sampled. This method is therefore recommended when responses are known to be linear or close to linear, and also when computational cost is high and rough estimates are acceptable.

**Second Order Taylor’s Expansion.** Adding the second order terms, the Taylor’s series expansion for a performance response, $Y$, is:

$$Y(x) = y + \frac{dY}{dx} \Delta x + \frac{1}{2} \Delta x^T \frac{d^2Y}{dx^2} \Delta x$$  \[11\]

The mean of this performance response is then obtained by taking the expectation of both sides of the expansion:

$$\mu_i = Y(\mu_i) + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 Y}{\partial x_i \partial x_j} \sigma_{x_i} \sigma_{x_j}$$  \[12\]

and the standard deviation of $Y(x)$ is given by:

$$\sigma_i = \left( \sum_{i=1}^{n} \left( \frac{\partial Y}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial^2 Y}{\partial x_i \partial x_j} \right)^2 \sigma_{x_i} \sigma_{x_j} \right)^{1/2}$$  \[13\]

where $\sigma_{x_i}$ is the standard deviation of the $i$th parameter and $\sigma_{x_j}$ is the standard deviation of the $j$th parameter. For more details on this approach for variability estimation and its use for robust design, see Ref. 25.

Since second order derivatives of responses, including crossed terms, with respect to random variables are needed in Eqn. 13, and the mean value point is needed in Eqn. 12, the second order Taylor’s expansion estimates require $(n+1)(n+2)/2$ analyses for evaluation. Consequently, this approach is significantly more computationally expensive than the first order Taylor’s expansion, and usually will be more expensive than DOE. For low numbers of uncertain parameters, this approach can still be more efficient than Monte Carlo simulation, but becomes less efficient with increasing $n$ (Monte Carlo simulation is not dependent on the number of parameters). For efficiency in implementing the second order Taylor’s expansion approach, the cross terms of Eqn. 13 are often ignored, and only pure second order terms are included (diagonal terms). The number of necessary evaluations is then reduced to $2n+1$. The second order Taylor’s expansion is recommended when curvature exists and the number of uncertain parameters is not excessive.

Any of the sampling methods summarized in this section can be used to estimate performance variability. These estimates are then used within the six sigma probabilistic optimization formulation to improve design quality.

**4 EXAMPLE: ROBUST, RELIABLE DESIGN OF A WELDED JOINT**

The six sigma based probabilistic design optimization implementation presented in the previous section is demonstrated in this section using a structural design problem, the design of a welded joint. In this example problem, the effects of optimizing for six sigma quality are clearly demonstrated. The welded joint is first optimized deterministically. The “quality” of the deterministic “optimum” is then measured using a six sigma probabilistic analysis, and improved using the six sigma probabilistic optimization implementation.

The welded joint configuration to be “optimized for quality” is illustrated in Figure 4. A plate is to be welded to a wall surface, and extends beyond the wall surface by 500 mm. A force of 10kN is applied at the cantilevered end of the plate. The welded joint is to be designed for minimum cost subject to constraints on shear stress of the weld bead, and bending stress, vertical deflection, and buckling strength of the plate.

![Figure 4 Welded Joint Configuration](image-url)
The following four design parameters define the welded joint configuration design space:

\[ \begin{align*}
2.0 \leq t &- \text{size of the bead} \leq 6.0 \text{ mm} \\
100.0 \leq s &- \text{length of the bead} \leq 500.0 \text{ mm} \\
100.0 \leq h &- \text{height of the plate} \leq 500.0 \text{ mm} \\
5.0 \leq b &- \text{thickness of the plate} \leq 10.0 \text{ mm}
\end{align*} \]

The deterministic optimization formulation for this problem is thus given as follows:

**Minimize:** Cost

**Subject to:**
- Max. Bead Shear Stress \( \leq 70.0 \text{ MPa} \)
- Max. Plate Bending Stress \( \leq 210 \text{ MPa} \)
- Max. Vertical Plate Deflection \( \leq 5.0 \text{ mm} \)
- Plate Buckling Strength \( \text{Pc: } 1-\text{Pc}/\text{P} \leq 0.0 \)
- Geometric Constraint \((t \leq b): t/b-1.0 \leq 0.0\)
- Design variable bounds (above)

The cost of the welded joint is measured using a metric that incorporates both the cost of the plate, due to changes in the cross section, and the cost of the weld due to changes in the size and length of the bead. The buckling strength of the plate must be no less than the applied load \( P \) of 10 kN shown in Figure 4, and the size of the bead, \( t \), must be less than the plate thickness, \( b \).

The baseline configuration for the welded joint, given in Table 2, is infeasible with respect to the maximum allowed vertical deflection of the plate (displayed in bold in Table 2). Consequently, deterministic optimization is performed first to identify a feasible design from which to begin probabilistic design analysis and optimization for quality measurement and improvement. The deterministic optimization solution is also given in Table 2.

To minimize cost, the optimizer reduces all four design variables as much as possible subject to satisfying the constraints. With the resulting “optimal” design, the cost is reduced by over 50%. However, this design has three active constraints, shown in bold in Table 2: bead shear stress, plate deflection, and plate buckling strength.

To analyze the quality of this design, the following eight parameters are defined as random variables, all normally distributed with coefficient of variation of 1%:

\[ \begin{align*}
t &- \text{size of the bead} \\
s &- \text{length of the bead} \\
h &- \text{height of the plate} \\
b &- \text{thickness of the plate} \\
P &- \text{applied load} \\
L &- \text{distance to wall} \\
E &- \text{Young’s modulus} \\
G &- \text{shear modulus}
\end{align*} \]

Given these random variables, a probabilistic design analysis is performed at the deterministic optimization solution by implementing Monte Carlo Simulation using descriptive sampling with 5000 points. The design quality results are provided in Table 2 along side the optimization design solution. In this column, quality results are given both as sigma level and percent reliability. As can be seen, the active constraints lead to a “sigma-level” around 0.7, or roughly 50% reliability (for example, sigma level of 0.693, reliability of 51.1% for bead shear stress). This active constraint quality level is displayed pictorially in the graphs of Figure 5 for bead shear stress (a) and buckling strength (b). Since the design is at the upper bound in both cases, the entire right half of the distributions are outside the constraint boundary.

![Quality Level for Shear Stress](attachment:Quality_Level_for_Shear_Stress.png)

**Figure 5.** Constraint Quality For Deterministic Optimization Solution

(a) Bead Shear Stress Constraint Quality

(b) Buckling Strength Constraint Quality
Notice in Figure 5 that the overall quality or sigma level displayed is different than the sigma position of the upper bound. Even though the upper bound is essentially zero-sigmas from the mean for these active constraints, the quality level is not zero sigma. With only one bound present, “sigma level” is a metric that reflects the percent variation outside the bound, the probability of failure, and the equivalent ±σ; with 50% of the distribution variation outside the constraint, the equivalent sigma level is ±0.7σ.

Notice also in Table 2 that the quality of the inputs is presented. In this case the quality is measured with respect to the design variable bounds. Since bead length, s, is driven to its lower bound, the sigma level is again ~0.7 and the reliability is 50%; this design parameter input bound quality is illustrated in Figure 6. While input design variable bounds may often be less “hard” requirements than output constraints, and so this quality level and potential for violation may not be as severe, the quality of these input bounds can easily be measured while measuring the output constraint quality.

In Figure 7, the variability of the objective, cost, is displayed in a probability distribution frequency graph. With the relatively low coefficient of variation of only 1% for the random variables, the standard deviation of cost is fairly low; the coefficient of variation of cost at this design is 1.64% as listed in Table 2. This leads to roughly ±5% potential variation in actual cost.

To improve the overall quality of the welded joint configuration, the six sigma probabilistic optimization formulation, as discussed in Section 3, is implemented. This formulation is given as follows:

**Minimize**

\[ \mu_{\text{Cost}} + \sigma_{\text{Cost}} + \sigma_{\text{BeadShearStress}} + \sigma_{\text{PlateBendingStress}} + \sigma_{\text{PlateDeflection}} + \sigma_{\text{Platebuckling}} + \sigma_{\text{GeometricConstraint}} \]

**Subject to**

**Output Quality Constraints:**

\[ \mu_{\text{BeadShearStress}} + 6\sigma_{\text{BeadShearStress}} \leq 70.0 \text{ MPa} \]
\[ \mu_{\text{PlateBendingStress}} + 6\sigma_{\text{PlateBendingStress}} \leq 210 \text{ MPa} \]
\[ \mu_{\text{PlateDeflection}} + 6\sigma_{\text{PlateDeflection}} \leq 5.0 \text{ mm} \]
\[ \mu_{\text{Platebuckling}} + 6\sigma_{\text{Platebuckling}} \leq 0.0 \]
\[ \mu_{\text{GeometricConstraint}} + 6\sigma_{\text{GeometricConstraint}} \leq 0.0 \]

**Input Quality Bounds:**

\[ 2.0 \leq \mu_{\text{b-BeadSize}} \leq 6.0 \text{ mm} \]
\[ 100.0 \leq \mu_{\text{s-BeadLength}} \leq 500.0 \text{ mm} \]
\[ 100.0 \leq \mu_{\text{b-PlateHeight}} \leq 500.0 \text{ mm} \]
\[ 5.0 \leq \mu_{\text{b-PlateThickness}} \leq 0.0 \]

The objective has been expanded from “minimize cost” to include not only minimizing both mean cost and cost variation (standard deviation of cost), but also includes the standard deviation of all output constraint parameters to seek robust designs. The output constraints are reformulated as quality constraints so that the mean plus six standard deviations is within the constraint upper bound for all outputs. In the case of input design variable bounds, with both lower and upper bounds defined, these input constraints are also reformulated as quality constraints by ensuring that the mean minus six standard deviations is within the lower bound, and the mean plus six standard deviations is within the upper bounds, if possible, for all design variables. Thus the quality goal in this formulation is “six sigma” quality level.

The results from implementing this six sigma formulation, using the second order Taylor’s expansion method for variability estimation, on the welded joint configuration are listed in Table 2 in the last two columns (nominal design solution and quality levels achieved). As expected, the design variables are all increased, both to satisfy the input bound quality of six sigma for bead length, s, and plate thickness, b, and to reduce the bead shear stress, the plate deflection, and the buckling constraint values to achieve the output constraint quality of six sigma for these previously
active constraints. For the new design, all input bounds and output constraints have a quality level of at least 6.0 as desired; compare Figure 8 displaying the bead length input bound quality for the six sigma design to Figure 6 for the deterministic optimum, and compare Figures 9 (a) and (b) for bead shear stress and buckling strength quality for the six sigma solution to Figures 5 (a) and (b) for the deterministic solution.

![Figure 8 Bead Length Design Variable Bound Quality For Six Sigma Solution](image)

The trade-off of increasing the quality level is an increase in cost. For the six sigma design, the nominal cost is increased by 30% over the deterministic optimum solution, but is still less than the cost of the baseline design. In this problem, the six sigma quality constraints drive the design, leaving little room for improvement with respect to the robust objective formulation. Since the coefficient of variation for the random variables is fixed at 1%, and the design variable values are increased to achieve the quality constraints, the standard deviation of the random design variables is increased, and the standard deviation of cost is increased with the increase in the mean cost value. The coefficient of variation for cost is slightly less for the six sigma solution, but this is obviously due to the increased mean cost value, since the standard deviation is increased as well; this coefficient of variation, and the cost standard deviation itself, is again quite low. The standard deviations of bead shear stress, plate bending stress, and plate deflection are decreased, as desired in the robust objective formulation.

### 5 SUMMARY AND CLOSING REMARKS

A six sigma based probabilistic design optimization formulation that combines concepts and approaches from structural reliability and robust design with current concepts and philosophy of six sigma is presented in this paper. To optimize for six sigma quality, uncertainty information and associated variability is incorporated in all aspects of this formulation, through the definition of random variables and their distributions and statistical properties, the definition of input quality constraints associated with random design variable bounds, the definition of output quality constraints, or reliability constraints, and the definition of a robust objective function for minimizing performance variation. This six sigma based probabilistic design optimization formulation is implemented within the iSIGHT design environment and is demonstrated in this paper for the structural design of a welded joint. The joint is designed for minimum cost and minimum variation, subject to achieving a six sigma quality level with respect to stress, deflection, buckling, and geometric constraints, and also with respect to input design variable bounds. A six sigma design is identified in which the increase in quality results in a tradeoff of increased cost compared to a deterministically optimized design.

The welded joint problem presented in this paper is a simple structural design example. While the quality concepts and probabilistic design are easily demonstrated, the concerns of computational expense of implementing probabilistic design and optimization procedures are not prevalent in this simple problem. For computationally expensive engineering analyses that can require hours or days for a single analysis, application of probabilistic design/optimization methods is not feasible. However, the incorporation of additional simulation based design technologies can support the application of probabilistic methods to complex, computationally expensive engineering design problems. Specifically, research in three areas has been making great strides in this direction: approximation methods, alternate sampling methods for variability estimation, and large scale parallel processing. Response surface approximation methods have been widely used in recent years to replace computationally expensive, high fidelity analysis codes with simple polynomial models that are essentially computationally free. More recently, approximation methods capable of more accurately modeling nonlinear responses, such as kriging, are being investigated for engineering design problems. New methods for variability estimation that are both efficient and sufficiently accurate are also continually being developed and refined. These methods are especially useful for verification of results obtained using approximate models; with good approximation models, computational expense during probabilistic optimization is not a concern, and thus Monte Carlo simulation is often the best sampling choice. Finally, with advances in parallel processing, given that sampling for variability estimation usually requires the execution of a number independent analyses, the time required to execute this large number of analyses for variability estimation is being radically reduced.

American Institute of Aeronautics and Astronautics
Figure 9 Constraint Quality For Six Sigma Solution

Table 2 Welded Joint Results: Optimization, Quality Analysis, Six Sigma Quality Improvement

<table>
<thead>
<tr>
<th>INPUTS</th>
<th>Baseline</th>
<th>Optimization</th>
<th>Six Sigma Probabilistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>t – bead size (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>µ</td>
<td>4.0</td>
<td>2.453</td>
<td>≥ 10 sigma</td>
</tr>
<tr>
<td>σ</td>
<td>0.0245</td>
<td>100% rel.</td>
<td>0.049</td>
</tr>
<tr>
<td>s – bead length (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>µ</td>
<td>300.0</td>
<td>100.0</td>
<td>0.674 sigma</td>
</tr>
<tr>
<td>σ</td>
<td>1.00</td>
<td>50.0% rel.</td>
<td>1.10</td>
</tr>
<tr>
<td>h – plate height (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>µ</td>
<td>300.0</td>
<td>290.2</td>
<td>≥ 10 sigma</td>
</tr>
<tr>
<td>σ</td>
<td>2.90</td>
<td>100% rel.</td>
<td>3.42</td>
</tr>
<tr>
<td>b – plate thickness (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>µ</td>
<td>7.5</td>
<td>5.039</td>
<td>1.226 sigma</td>
</tr>
<tr>
<td>σ</td>
<td>0.0504</td>
<td>78.0% rel.</td>
<td>0.053</td>
</tr>
</tbody>
</table>

| OUTPUTS           |          |              |                         |
| Cost Fnc.         |          |              |                         |
| µ                 | 5.608    | 2.617        | 1.64 %                  |
| σ                 | 0.0429   | Coeff. Var. | 0.0539                  |
| Bead Shear Stress ≤ 70.0 MPa | |              |                         |
| µ                 | 17.998   | 69.942       | 0.693 sigma             |
| σ                 | 1.340    | 51.1% rel.  | 0.539                   |
| Plate Bending Stress ≤ 210 MPa | |              |                         |
| µ                 | 44.444   | 70.674       | ≥ 10 sigma              |
| σ                 | 1.860    | 100% rel.   | 1.276                   |
| Plate Deflection ≤ 5.0 mm | |              |                         |
| µ                 | 6.719    | 4.985        | 0.702 sigma             |
| σ                 | 0.229    | 51.7% rel.  | 0.147                   |
| Plate Buckling Constraint ≤ 0.0 | |              |                         |
| µ                 | -2.377   | -0.0007      | 0.696 sigma             |
| σ                 | 0.0369   | 51.3% rel.  | 0.0483                  |
| Geometric Constraint ≤ 0.0 | |              |                         |
| µ                 | -0.467   | -0.513       | ≥ 10 sigma              |
| σ                 | 0.00689  | 100% rel.   | 0.0130                  |

REFERENCES


