MacCormack’s Techniques
Introduction

• Original (1969) method is 2nd order accurate (both in space and time), explicit method

• It is a modified form of the Lax-Wendroff scheme, but is much simpler in its applications

• For purposes of illustration, we now again address the Euler equations.

\[
\begin{align*}
\text{Continuity} & \quad \frac{\partial \rho}{\partial t} = -\left( \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} \right) \\
\text{x momentum} & \quad \frac{\partial u}{\partial t} = -\left( u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} \right) \\
\text{y momentum} & \quad \frac{\partial v}{\partial t} = -\left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} \right) \\
\text{Energy} & \quad \frac{\partial e}{\partial t} = -\left( \rho \frac{\partial e}{\partial x} + u \frac{\partial e}{\partial y} + p \frac{\partial u}{\partial x} + \frac{p}{\rho} \frac{\partial v}{\partial y} \right)
\end{align*}
\]
General expression

The following expression shows how density for the next time level is calculated using the calculated average value of the derivative

\[ \rho_{i}^{n+1} = \rho_{i}^{n} + \left( \frac{\partial \rho}{\partial t} \right)_{\text{avg}} \Delta t \] ........(6.13)

Similarly, relations for the other flow-field variables

\[ u_{i}^{n+1} = u_{i}^{n} + \left( \frac{\partial u}{\partial t} \right)_{\text{avg}} \] \[ \Delta t \] ........(6.14)

\[ v_{i}^{n+1} = v_{i}^{n} + \left( \frac{\partial v}{\partial t} \right)_{\text{avg}} \Delta t \] ........(6.15)

\[ e_{i}^{n+1} = e_{i}^{n} + \left( \frac{\partial e}{\partial t} \right)_{\text{avg}} \] \[ \Delta t \] ........(6.16)
Step 1: Predictor step

- In continuity equation, replace the spatial derivatives on the right-hand side with \textit{forward} differences

\[
\left( \frac{\partial \rho}{\partial t} \right)^t_{i,j} = -\left( \rho_{i,j}^t \frac{u^t_{i+1,j} - u^t_{i,j}}{\Delta x} + u^t_{i,j} \frac{\rho^t_{i+1,j} - \rho^t_{i,j}}{\Delta x} 
+ \rho_{i,j}^t \frac{v^t_{i+1,j} - v^t_{i,j}}{\Delta y} + v^t_{i,j} \frac{\rho^t_{i+1,j} - \rho^t_{i,j}}{\Delta y} \right)
\]

- In above equation, all flow variables at time \( t \) are known
Step 1: Predictor step

- Predicted value from the first two terms of a Taylor series

\[
\bar{\rho}_{i,j}^{t+\Delta t} = \rho_{i,j}^t + \left( \frac{\partial \rho}{\partial t} \right)_{i,j}^t \Delta t \quad (6.18)
\]

- It is only first order accurate

- Similar equations can be written for the predicted values of the other variables

\[
\bar{u}_{i,j}^{t+\Delta t} = u_{i,j}^t + \left( \frac{\partial u}{\partial t} \right)_{i,j}^t \Delta t \quad (6.19)
\]

\[
\bar{v}_{i,j}^{t+\Delta t} = v_{i,j}^t + \left( \frac{\partial v}{\partial t} \right)_{i,j}^t \Delta t \quad (6.20)
\]

\[
\bar{e}_{i,j}^{t+\Delta t} = e_{i,j}^t + \left( \frac{\partial e}{\partial t} \right)_{i,j}^t \Delta t \quad (6.21)
\]
Step 2: Corrector step

- We first obtain a predicted value of the time derivatives at time \( t + \Delta t \), by substituting the predicted value of \( \rho \), \( u \), and \( v \) into the right side of the continuity equation, replacing the spatial derivative with rearward differences

\[
\left( \frac{\partial \rho}{\partial t} \right)_{i,j}^{t+\Delta t} = - \left( \rho_{i,j}^{t+\Delta t} \frac{u_{i,j}^{t+\Delta t} - u_{i-1,j}^{t+\Delta t}}{\Delta x} + u_{i,j}^{t+\Delta t} \frac{\rho_{i,j}^{t+\Delta t} - \rho_{i-1,j}^{t+\Delta t}}{\Delta x} \right) + \left( \rho_{i,j}^{t+\Delta t} \frac{v_{i,j}^{t+\Delta t} - v_{i,j-1}^{t+\Delta t}}{\Delta y} + v_{i,j}^{t+\Delta t} \frac{\rho_{i,j}^{t+\Delta t} - \rho_{i,j-1}^{t+\Delta t}}{\Delta y} \right)
\]

\((6.21)\)
Step 3: Average Value

- Now find average value of the time derivative as

\[
\left( \frac{\partial \rho}{\partial t} \right)_{avg} = \frac{1}{2} \left[ \left( \frac{\partial \rho}{\partial t} \right)_{i,j} + \left( \frac{\partial \rho}{\partial t} \right)_{t+\Delta t} \right]
\]

From Eq. (6.17), predictor

\[
\left( \frac{\partial \rho}{\partial t} \right)_{i,j} = -\left( \rho_{i,j}^t \frac{u_{i+1,j}^t - u_{i,j}^t}{\Delta x} + \rho_{i+1,j}^t - \rho_{i,j}^t \right)
\]

\[
+ \rho_{i,j}^t \frac{v_{i+1,j}^t - v_{i,j}^t}{\Delta y} + v_{i,j}^t \left( \rho_{i+1,j}^t - \rho_{i,j}^t \right)
\]

From Eq. (6.21), corrector

\[
\left( \frac{\partial \rho}{\partial t} \right)_{i,j} = -\left( \bar{\rho}_{i,j}^{t+\Delta t} \frac{\bar{u}_{i,j}^{t+\Delta t} - \bar{u}_{i-1,j}^{t+\Delta t}}{\Delta x} + \bar{u}_{i,j}^{t+\Delta t} \frac{\bar{\rho}_{i,j}^{t+\Delta t} - \bar{\rho}_{i-1,j}^{t+\Delta t}}{\Delta x} \right)
\]

\[
+ \bar{\rho}_{i,j}^{t+\Delta t} \frac{\bar{v}_{i,j}^{t+\Delta t} - \bar{v}_{i,j-1}^{t+\Delta t}}{\Delta y} + \bar{v}_{i,j}^{t+\Delta t} \left( \bar{\rho}_{i,j}^{t+\Delta t} - \bar{\rho}_{i,j-1}^{t+\Delta t} \right)
\]
Step 4: Value at time $t+\Delta t$

• Corrected value of the density at time $t+\Delta t$

$$\bar{\rho}_{i,j}^{t+\Delta t} = \rho_{i,j}^t + \left( \frac{\partial \rho}{\partial t} \right)_{AV}^t \Delta t$$

• The predictor-corrector sequence is repeated at all grid points to obtain the density throughout the flow field at time $t+\Delta t$.

• We can use the same technique to calculate $u,v,$ and $e$ at time $t+\Delta t$. 
Summary of MacCormack’s Technique

• The method has a predictor step and a corrector step.
• An average value of the time derivative is first calculated.
• Average value of the time derivative are calculated by using a predictor step and a corrector step.
• Solution is advanced to the next time level using the average value of the time derivative.
• Both forward and backward differences are used for space derivatives in calculating the average value of the time derivative.
  – The reason is for second order accuracy.
Discussion on MacCormack’s Technique

• Because of using forward difference for the predictor and backward difference for the corrector steps, the method has 2\textsuperscript{nd} order accuracy as the Lax-Wendroff method.

• But it is much easier to apply, because there is no need to evaluate the second time derivatives.

• The MacCormack’s technique can be used for solutions of the unsteady N-S equations by means of time-marching solutions as we have done for solutions of Euler equations.
Example

• Problem
  – Investigate the subsonic compressible flow over the Wotmann airfoil.
  – What are differences between laminar and turbulent flow over this airfoil for Re= 100,000 ?

• Governing equations
  – 2-D N-S equations for viscous flow

• Numerical technique
  – The MacCormack’s technique
Example

Laminar flow & Turbulent flow

(a) Laminar flow

(b) Turbulent flow
Example

- Lift coefficient versus angle of attack for a Wormann airfoil
  \( \text{Re}=100,000; \ M=0.5 \)

- The laminar flow value of \( c_l \) is not even close to the experiment measurement
- The turbulent value of \( c_l \) is in close agreement with the experiment data.
Example

- Drag coefficient versus angle of attack for a Wormann airfoil
  \( Re=100,000; M=0.5 \)

- For the laminar flow value, the agreement with experiment is poor.
- The turbulent value of \( c_l \) is in close agreement with the experiment data.